

Single Letter Expression of Capacity for a Class of Channels with Memory

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Abstract

We study finite alphabet channels with Unit Memory on the previous Channel Outputs called UMCO channels. We identify necessary and sufficient conditions, to test whether the capacity achieving channel input distributions with feedback are time-invariant, and whether feedback capacity is characterized by single letter, expressions, similar to that of memoryless channels. The method is based on showing that a certain dynamic programming equation, which in general, is a nested optimization problem over the sequence of channel input distributions, reduces to a non-nested optimization problem. Moreover, for UMCO channels, we give a simple expression for the ML error exponent, and we identify sufficient conditions to test whether feedback does not increase capacity. We derive similar results, when transmission cost constraints are imposed. We apply the results to a special class of the UMCO channels, the Binary State Symmetric Channel (BSSC) with and without transmission cost constraints, to show that the optimization problem of feedback capacity is non-nested, the capacity achieving channel input distribution and the corresponding channel output transition probability distribution are time-invariant, and feedback capacity is characterized by a single letter formulae, precisely as Shannon's single letter characterization of capacity of memoryless channels. Then we derive closed form expressions for the capacity achieving channel input distribution and feedback capacity. We use the closed form expressions to evaluate an error exponent for ML decoding.

I. INTRODUCTION

Shannon in his landmark paper [1], showed that the capacity of Discrete Memoryless Channels (DMCs) $\{\mathbb{A}, \mathbb{B}, \{\mathbf{P}_{B|A}(b|a) : (a, b) \in \mathbb{A} \times \mathbb{B}\}\}$ is characterized by the celebrated single letter formulae

$$C \triangleq \max_{\mathbf{P}_A} I(A; B). \quad (\text{I.1})$$

This is often shown by using the converse to the channel coding theorem, to obtain the upper bounds [2]

$$C_{A^n; B^n}^{noFB} \triangleq \max_{\mathbf{P}_{A^n}} I(A^n; B^n) \leq \max_{\mathbf{P}_{A_i, i=0, \dots, n}} \sum_{i=0}^n I(A_i; B_i) \leq (n+1)C \quad (\text{I.2})$$

which are achievable, if and only if the channel input distribution satisfies conditional independence $\mathbf{P}_{A_i|A^{i-1}} = \mathbf{P}_{A_i}, i = 0, 1, \dots, n$, and $\{A_i : i = 0, 1, \dots, n\}$ is identically distributed, which implies that the joint process $\{(A_i, B_i) : i = 0, 1, \dots, n\}$ is independent and identically distributed, and hence stationary ergodic. For DMCs, it is shown by Shannon [3] and Dobrushin [4] that feedback codes do not incur a higher capacity compared to that of codes without feedback, that is, $C^{FB} = C$. This is often shown by first applying the converse to the coding theorem, to deduce that feedback does not increase capacity [5], that is, $C^{FB} \leq C_{A^n; B^n}^{noFB} = C$, which then implies that any candidate of optimal channel input distribution with feedback $\{\mathbf{P}_{A_i|A^{i-1}, B^{i-1}} : i = 0, \dots, n\}$ satisfies conditional independence

$$\mathbf{P}_{A_i|A^{i-1}, B^{i-1}}(da_i|a^{i-1}, b^{i-1}) = \mathbf{P}_{A_i}(da_i), \quad i = 0, 1, \dots, n \quad (\text{I.3})$$

and hence identity $C^{FB} = C^{noFB} = C$ holds if $\{A_i : i = 0, 1, \dots\}$ is identically distributed.

For general channels with memory defined by $\{\mathbf{P}_{B_i|B^{i-1}, A_i} : i = 0, 1, \dots, n\}$, $\mathbf{P}_{B_0|B^{-1}, A_0} = \mathbf{P}_{B_0|B^{-1}, A_0}$, where B^{-1} is the initial state, in general, feedback codes incur a higher capacity than codes without feedback [2], [6]. The information measure often employed to characterize feedback capacity of such channels is Marko's directed information [7], put forward by Massey [8], and defined by

$$I(A^n \rightarrow B^n) = \sum_{i=0}^n I(A^i; B_i | B^{i-1}) \triangleq \sum_{i=0}^n \int \log \left(\frac{\mathbf{P}_{B_i|B^{i-1}, A^i}(\cdot | b^{i-1}, a^i)}{\mathbf{P}_{B_i|B^{i-1}}(\cdot | b^{i-1})} (b_i) \right) \mathbf{P}_{A^i, B^i}(da^i, db^i). \quad (I.4)$$

Indeed, Massey [8] showed that the per unit time limit of the supremum of directed information over channel input distributions $\mathcal{P}_{[0,n]}^{FB} \triangleq \{\mathbf{P}_{A_i|A^{i-1}, B^{i-1}} : i = 0, \dots, n\}$, defined by

$$C_{A^\infty \rightarrow B^\infty}^{FB} = \lim_{n \rightarrow \infty} \frac{1}{n+1} C_{A^n \rightarrow B^n} \quad C_{A^n \rightarrow B^n} \triangleq \sup_{\mathcal{P}_{[0,n]}^{FB}} I(A^n \rightarrow B^n) \quad (I.5)$$

gives a tight bound on any achievable rate of feedback codes, and hence $C_{A^\infty \rightarrow B^\infty}^{FB}$ is a candidate for the capacity of feedback codes. However, for channels with memory, it is generally not known whether the multi-letter expression of capacity, (I.5), can be reduced to a single letter expression, analogous to (I.1).

Our main objective is to provide a framework for a single letter characterization of feedback capacity for a general class of channels with memory. Towards this direction, we provide conditions on channels with memory such that

$$C_{A^n \rightarrow B^n}^{FB} = (n+1)C^{FB} \quad (I.6)$$

where C^{FB} is a single letter expression similar to that of DMCs. Specifically, for channels of the form $\{\mathbf{P}_{B_i|B_{i-1}, A_i} : i = 0, 1, \dots, n\}$, where $B_{-1} = b_{-1} \in \mathbb{B}_{-1}$ is the initial state, we give necessary and sufficient conditions such that the following equality holds.

$$C_{A^n \rightarrow B^n}^{FB} = (n+1) \sup_{\mathbf{P}_{A_0|B_{-1}}(\cdot | b_{-1})} I(A_0; B_0 | b_{-1}), \quad \forall b_{-1} \in \mathbb{B}_{-1}. \quad (I.7)$$

That is, the single letter expression is $C^{FB} \triangleq \sup_{\mathbf{P}_{A_0|B_{-1}}(\cdot | b_{-1})} I(A_0; B_0 | b_{-1})$, and is independent of the initial state $b_{-1} \in \mathbb{B}_{-1}$.

A. Main Results and Methodology

First, we consider channels with Unit Memory on the previous Channel Output (UMCO), defined by

$$\mathbf{P}_{B_i|B^{i-1}, A^i} = \mathbf{P}_{B_i|B_{i-1}, A_i}, \quad i = 0, 1, \dots, n \quad (I.8)$$

with and without a transmission cost constraint defined by

$$\frac{1}{n+1} \mathbf{E} \left\{ \sum_{i=0}^n \gamma_i^{UM}(A_i, B_{i-1}) \right\} \quad (I.9)$$

where $\gamma_i^{UM} : \mathbb{A}_i \times \mathbb{B}_{i-1} \mapsto [0, \infty)$. We identify necessary and sufficient conditions on the channel so that the optimization problem $C_{A^n \rightarrow B^n}^{FB}$, which is generally a nested optimization problem, often dealt with via dynamic programming, reduces to a non-nested optimization problem. These conditions give rise to a single letter characterization of feedback capacity. Among other results, we derive sufficient conditions for feedback not to increase capacity, and identify sufficient conditions for asymptotic stationarity of optimal channel input distribution and ergodicity of the joint process $\{(A_i, B_i) : i = 0, 1, \dots\}$. Moreover, we give an upper bound on the error probability of maximum likelihood decoding. We also treat problems with transmission cost constraints.

Second, we apply the framework of the UMCO channel on the Binary State Symmetric Channel (BSSC),

defined by

$$\mathbf{P}_{B_i|A_i, B_{i-1}}(b_i|a_i, b_{i-1}) = \begin{matrix} & \begin{matrix} 0,0 & 0,1 & 1,0 & 1,1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{bmatrix} \alpha & \beta & 1-\beta & 1-\alpha \\ 1-\alpha & 1-\beta & \beta & \alpha \end{bmatrix} \end{matrix}, \quad i = 0, 1, \dots, n, \quad (\alpha, \beta) \in [0, 1] \times [0, 1] \quad (\text{I.10})$$

with and without a transmission cost constraint defined by

$$\frac{1}{n+1} \mathbf{E} \left\{ \sum_{i=0}^n \gamma(A_i, B_{i-1}) \right\} \leq \kappa, \quad \gamma(a_i, b_{i-1}) = \overline{a_i \oplus b_{i-1}}, \quad \kappa \in [0, \kappa_{\max}] \quad (\text{I.11})$$

where $\overline{x \oplus y}$ denotes the compliment of the modulo2 addition of x and y . We calculate the capacity achieving channel input distribution with feedback without cost constraint and show that it is time-invariant. This illustrates that feedback capacity satisfies (I.6), it is independent of the initial state $B_{-1} = b_{-1}$, and it is characterized by

$$C_{A^\infty \rightarrow B^\infty}^{FB} = \sup_{\mathbf{P}_{A_0|B_{-1}}} I(A_0; B_0|b_{-1}), \quad \forall b_{-1} \in \mathbb{B}_{-1} \quad (\text{I.12})$$

$$= H(\lambda) - \nu H(\alpha) - (1-\nu)H(\beta) \quad (\text{I.13})$$

where λ, ν are functions of channel parameters α, β (see Theorem IV.1). The characterization (I.12) is precisely analogous to the single letter characterization of (I.1) and (I.2) of capacity of DMCs. Additionally, we provide the error exponent evaluated on the capacity achieving channel input distribution with feedback, and we derive an upper bound on the error probability of maximum likelihood decoding which is easy to compute (see Section IV-A3). Finally, we show that a time-invariant first order Markov channel input distribution without feedback achieves feedback capacity (I.13), and we give the closed form expressions both for the capacity achieving channel input distribution and the corresponding channel output distribution. We also treat the case with cost constraint.

The main mathematical concept we invoke to obtain the above results are the structural properties of the optimal channel input distributions, [9], [10]. Specifically the following.

- (a) For channels with infinite memory on the previous channel outputs defined by $\mathbf{P}_{B_i|B^{i-1}, A^i} = \mathbf{P}_{B_i|B^{i-1}, A_i}$, the maximization of directed information $I(A^n \rightarrow B^n)$ occurs in the subset satisfying conditional independence $\{\mathbf{P}_{A_i|A^{i-1}, B^{i-1}} = \mathbf{P}_{A_i|B^{i-1}} : i = 0, \dots, n\}$.
- (b) For channels with limited memory of order M defined by $\mathbf{P}_{B_i|B^{i-1}, A^i} = \mathbf{P}_{B_i|B_{i-M}^{i-1}, A_i}$, the maximization of directed information $I(A^n \rightarrow B^n)$ occurs in the subset satisfying conditional independence $\{\mathbf{P}_{A_i|A^{i-1}, B^{i-1}} = \mathbf{P}_{A_i|B_{i-M}^{i-1}} : i = 0, \dots, n\}$.
- (c) For the UMCO channel the maximization of directed information $I(A^n \rightarrow B^n)$ occurs in the subset satisfying conditional independence $\{\mathbf{P}_{A_i|A^{i-1}, B^{i-1}} = \mathbf{P}_{A_i|B_{i-1}} : i = 0, \dots, n\}$.

The structural properties, (a), (b) and (c), along with the fact that $C_{A^n \rightarrow B^n}^{FB} \geq C_{A^n, B^n}^{noFB}$, are employed in Section II to provide sufficient conditions for feedback not to increase the capacity. Moreover, the structural property of the UMCO channel, (c), is applied in Section III to construct the finite horizon dynamic programming, the necessary and sufficient conditions on the capacity achieving input distribution, and the necessary and sufficient conditions for the non-nested optimization of feedback capacity. The methodology and the corresponding theorems of Section III can be easily extended to channels with finite memory on previous channel outputs by invoking the structural properties of the capacity achieving distributions for these channels.

B. Relation to the Literature

Although for several years significant effort has been devoted to the study of channels with memory, with or without feedback, explicit or closed form expressions for capacity of such channels are limited

to few but ripe cases. For non-stationary non-ergodic Additive Gaussian Noise (AGN) channels with memory, Cover and Pombra [5] showed that feedback codes can increase capacity by at most half a bit. On the other hand, for a finite alphabet version of the Cover and Pombra channel with certain symmetry, Alajaji [11] showed that feedback does not increase capacity. Moreover, Permuter, Cuff, Van Roy and Weissman [12] derived the feedback capacity of the trapdoor channel, while Elishco and Permuter [13] employed dynamic programming to evaluate feedback capacity of the Ising channel.

The capacity of channels $\{\mathbf{P}_{B_i|B_{i-1},A_i} : i = 0, \dots, n\}$ for feedback codes is analyzed by Berger [14] and Chen and Berger [15], under the assumption that the capacity achieving distribution satisfies conditional independence property $\mathbf{P}_{A_i|A^{i-1},B^{i-1}} = \mathbf{P}_{A_i|B_{i-1}}(a_i|b_{i-1}), i = 0, 1, \dots, n$. A derivation of this structural property of capacity achieving distribution is given in [9], [10].

Recently, Permuter, Asnani and Weissman [16], [17] derived the feedback capacity for a Binary-Input Binary-Output (BIBO) channel, called the Previous Output State (POST) channel, where the current state of the channel is the previously received symbol. The authors in [17], showed, among other results, that feedback does not increase capacity. It can be shown that the POST channel is within a transformation equivalent to the Binary State Symmetric channel (BSSC) [18], in which the state of the channel is defined as the modulo2 addition of the current input symbol and the previous output symbol. When there are no transmission cost constraints, our results for the BSSC compliment existing results obtained in [16], [17] regarding the POST channel, in the sense that, we show the time-invariant properties of the capacity achieving distributions, which implies the single letter characterization of feedback capacity, we derive closed form expressions for these distributions, provide an upper bound on the error probability of maximum likelihood decoding, and we show that a first-order Markov channel input distribution without feedback achieves feedback capacity. Moreover, we derive similar closed form expressions when averaged transmission cost constraints are imposed.

A portion of the results established in this paper were utilized to construct a Joint Source Channel Coding (JSCC) scheme for the BSSC with a cost constraint and the Binary Symmetric Markov Source (BSMS) with single letter Hamming distortion measure [19]. The scheme is a natural generalization of the JSCC design (uncoded transmission) of an Independent and Identically Distributed (IID) Bernoulli source over a Binary Symmetric Channel (BSC) [20], [21].

The remainder of the paper is organized as follows. In Section II, we introduce the mathematical formulation and identify sufficient conditions for feedback not to increase capacity. In Section III, we identify sufficient conditions to test whether the capacity achieving input distribution is time invariant. The results are then extended to the infinite horizon case. In Section IV, we apply the main theorems of section III to the BSSC, with and without feedback and with and without cost constraint, to prove, among other results, that capacity is given by a single letter characterization. Finally, Section V delivers our concluding remarks.

II. FORMULATION & PRELIMINARY RESULTS

In this section we introduce the definitions of feedback capacity, capacity without feedback, and we identify necessary and sufficient conditions for feedback not to increase the capacity.

A. Notation and Definitions

The probability distribution of a Random Variable (RV) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by the mapping $X : (\Omega, \mathcal{F}) \mapsto (\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is denoted by $\mathbf{P}(\cdot) \equiv \mathbf{P}_X(\cdot)$. The space of probability distributions on \mathbb{X} is denoted by $\mathcal{M}(\mathbb{X})$. A RV is called discrete if there exists a countable set \mathcal{S} such that $\sum_{x_i \in \mathcal{S}} \mathbb{P}\{\omega \in \Omega : X(\omega) = x_i\} = 1$. The probability distribution $\mathbf{P}_X(\cdot)$ is then concentrated on points in \mathcal{S} , and it is defined by

$$\mathbf{P}_X(A) \triangleq \sum_{x_i \in \mathcal{S} \cap A} \mathbb{P}\{\omega \in \Omega : X(\omega) = x_i\}, \quad \forall A \in \mathcal{B}(\mathbb{X}). \quad (\text{II.14})$$

Given another RV $Y : (\Omega, \mathcal{F}) \mapsto (\mathbb{Y}, \mathcal{B}(\mathbb{Y}))$, $\mathbf{P}_{Y|X}(dy|x)(\omega)$ is the conditional distribution of RV Y given X . For a fixed $X = x$ we denote the conditional distribution by $\mathbf{P}_{Y|X}(dy|X = x) = \mathbf{P}_{Y|X}(dy|x)$.

Let \mathbb{Z} denote the set of integers and $\mathbb{N} \triangleq \{0, 1, 2, \dots\}$, $\mathbb{N}^n \triangleq \{0, 1, 2, \dots, n\}$. The channel input and channel output spaces are sequences of measurable spaces $\{(\mathbb{A}_i, \mathcal{B}(\mathbb{A}_i)) : i \in \mathbb{Z}\}$ and $\{(\mathbb{B}_i, \mathcal{B}(\mathbb{B}_i)) : i \in \mathbb{Z}\}$, respectively, while their product spaces are $\mathbb{A}^{\mathbb{Z}} \triangleq \times_{i \in \mathbb{Z}} \mathbb{A}_i$, $\mathbb{B}^{\mathbb{Z}} \triangleq \times_{i \in \mathbb{Z}} \mathbb{B}_i$, $\mathcal{B}(\mathbb{A}^{\mathbb{Z}}) \triangleq \otimes_{i \in \mathbb{Z}} \mathcal{B}(\mathbb{A}_i)$, $\mathcal{B}(\mathbb{B}^{\mathbb{Z}}) \triangleq \otimes_{i \in \mathbb{Z}} \mathcal{B}(\mathbb{B}_i)$. Points in the product spaces are denoted by $a^n \triangleq \{\dots, a_{-1}, a_0, a_1, \dots, a_n\} \in \mathbb{A}^n$ and $b^n \triangleq \{\dots, b_{-1}, b_0, b_1, \dots, b_n\} \in \mathbb{B}^n, n \in \mathbb{Z}$.

B. Capacity with Feedback & Properties

Next, we provide the precise formulation of information capacity and some preliminary results. We begin by introducing the definitions of channel distribution, channel input distribution, transmission cost constraint, and feedback code.

Definition II.1. (Channel distribution with memory)

A sequence of conditional distributions defined by

$$\mathcal{C}_{[0,n]} \triangleq \left\{ \mathbf{P}_{B_i|B^{i-1}, A^i}(db_i|b^{i-1}, a^i) = \mathbf{P}_{B_i|B^{i-1}, A_i}(db_i|b^{i-1}, a_i) : i = 0, \dots, n \right\}. \quad (\text{II.15})$$

At time $i = 0$ the conditional distribution is $\mathbf{P}_{B_0|B^{-1}, A_0}(db_0|b^{-1}, a_0)$, where $B^{-1} = b^{-1} \in \mathbb{B}^{-1}$ is the initial data.

The initial data, $b^{-1} \in \mathbb{B}^{-1}$, denotes the initial state of the channel and this should not be misinterpreted as feedback information. In this work we assume that the initial data are known both to the encoder and the decoder, unless we state otherwise.

Definition II.2. (Channel input distribution with feedback)

A sequence of conditional distributions defined by

$$\mathcal{P}_{[0,n]}^{FB} \triangleq \left\{ \mathbf{P}_{A_i|A^{i-1}, B^{i-1}}(da_i|a^{i-1}, b^{i-1}) : i = 0, \dots, n \right\}. \quad (\text{II.16})$$

At time $i = 0$ the conditional distribution is $\mathbf{P}_{A_0|A^{-1}, B^{-1}}(da_0|a^{-1}, b^{-1}) = \mathbf{P}_{A_0|B^{-1}}(da_0|b^{-1})$. That is, the information structure of the channel input distribution is $\mathcal{I}_i^{FB} \triangleq \{b^{-1}, a_0, b_0, a_1, b_1, \dots, a_{i-1}, b_{i-1}\}$, for $i = 0, \dots, n$. For $i = 0$ the convention is $\mathcal{I}_0^{FB} \triangleq \{a^{-1}, b^{-1}\} = \{b^{-1}\}$, which states that the channel input distribution depends only on the initial data.

Definition II.3. (Transmission cost constraints)

The cost of transmitting symbols over the channel (II.15) is a measurable function $c_{0,n} : \mathbb{A}^n \times \mathbb{B}^{n-1} \mapsto [0, \infty)$ defined by

$$c_{0,n}(a^n, b^{n-1}) \triangleq \sum_{i=0}^n \gamma_i(a_i, b^{i-1}). \quad (\text{II.17})$$

The transmission cost constraint is defined by

$$\mathcal{P}_{[0,n]}^{FB}(\kappa) \triangleq \left\{ \mathbf{P}_{A_i|A^{i-1}, B^{i-1}}, i = 0, \dots, n : \frac{1}{n+1} \mathbf{E}_{\mu} \{c_{0,n}(A^n, B^{n-1})\} \leq \kappa \right\}, \quad \kappa \in [0, \infty] \quad (\text{II.18})$$

where $\kappa \in [0, \infty)$, and the subscript notation \mathbf{E}_{μ} indicates the joint distribution over which the expectation is taken is parametrized by the initial distribution $\mathbf{P}_{B^{-1}}(db^{-1}) = \mu(db^{-1})$ (and of course the channel input distribution).

Definition II.4. (Feedback code)

A feedback code for the channel defined by (II.15) with transmission cost constraint $\mathcal{P}_{[0,n]}^{FB}(\kappa)$ is a sequence $\{(n, M_n, \varepsilon_n) : n = 0, 1, \dots\}$, which consist of the following elements.

- (a) A set of uniformly distributed messages $\mathcal{M}_n \triangleq \{1, \dots, M_n\}$ and a set of encoding strategies, mapping messages into channel inputs of block length $(n+1)$, defined by¹

$$\begin{aligned} \mathcal{E}_{[0,n]}^{FB}(\kappa) &\triangleq \left\{ g_i : \mathcal{M}_n \times \mathbb{A}^{i-1} \times \mathbb{B}^{i-1} \mapsto \mathbb{A}_i, \quad a_0 = g_0(w, b^{-1}), a_1 = g_1(w, b^{-1}, a_0, b_0), \dots, \right. \\ &\left. a_n = g_n(w, b^{-1}, a_0, b_0, \dots, a_{n-1}, b_{n-1}), w \in \mathcal{M}_n : \quad \frac{1}{n+1} \mathbf{E}^g \left(c_{0,n}(A^n, B^{n-1}) \right) \leq \kappa \right\}, \quad n = 1, 2, \dots \end{aligned} \quad (\text{II.19})$$

The codeword for any $w \in \mathcal{M}_n$ is $u_w \in \mathbb{A}^n$, $u_w = (g_0(w, b^{-1}), g_1(w, b^{-1}, a_0, b_0), \dots, g_n(w, b^{-1}, a_0, b_0, \dots, a_{n-1}, b_{n-1}))$, and $\mathcal{C}_n = (u_1, u_2, \dots, u_{M_n})$ is the code for the message set \mathcal{M}_n , and $\{A^{-1}, B^{-1}\} = \{b^{-1}\}$. In general, the code depends on the initial data, depending on the convention, i.e., $B^{-1} = b^{-1}$, which are known to the encoder and decoder (unless specified otherwise). Alternatively, we can take $\{A^{-1}, B^{-1}\} = \{\emptyset\}$.

- (b) Decoder measurable mappings $d_{0,n} : \mathbb{B}^n \mapsto \mathcal{M}_n$, such that the average probability of decoding error satisfies

$$\mathbf{P}_e^{(n)} \triangleq \frac{1}{M_n} \sum_{w \in \mathcal{M}_n} \mathbf{P}^g \left\{ d_{0,n}(B^n) \neq w | W = w \right\} \equiv \mathbf{P}^g \left\{ d_{0,n}(B^n) \neq W \right\} \leq \varepsilon_n$$

and the decoder may also assume knowledge of the initial data.

The coding rate or transmission rate over the channel is defined by $r_n \triangleq \frac{1}{n+1} \log M_n$. A rate R is said to be an achievable rate, if there exists a code sequence satisfying $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ and $\liminf_{n \rightarrow \infty} \frac{1}{n+1} \log M_n \geq R$.

The operational definition of feedback capacity of the channel is the supremum of all achievable rates, i.e., $C \triangleq \sup \{R : R \text{ is achievable}\}$.

Given any channel input distribution $\{\mathbf{P}_{A_i|A^{i-1}, B^{i-1}} : i = 0, 1, \dots, n\} \in \mathcal{P}_{[0,n]}^{FB}$, a channel distribution $\{\mathbf{P}_{B_i|B^{i-1}, A_i} : i = 0, 1, \dots, n\}$, and a fixed initial distribution $\mu(b^{-1})$, then the induced joint distribution² \mathbf{P}_{A^n, B^n} parametrized by $\mu(\cdot)$ is uniquely defined, and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying the sequence of RVs $(A^n, B^n) \triangleq \{B^{-1}, A_0, B_0, A_1, B_1, \dots, A_n, B_n\}$ is constructed, as follows.

$$\begin{aligned} \mathbb{P}\{A^n \in da^n, B^n \in db^n\} &\triangleq \mathbf{P}_{A^n, B^n}(da^n, db^n) \\ &= \otimes_{j=0}^n \left(\mathbf{P}_{B_j|B^{j-1}, A_j}(db_j|b^{j-1}, a_j) \otimes \mathbf{P}_{A_j|A^{j-1}, B^{j-1}}(da_j|a^{j-1}, b^{j-1}) \right) \otimes \mu(db^{-1}). \end{aligned} \quad (\text{II.20})$$

$$\mathbb{P}\{B^n \in db^n\} \triangleq \mathbf{P}_{B^n}(db^n) = \int_{\mathbb{A}^n} \mathbf{P}_{A^n, B^n}(da^n, db^n). \quad (\text{II.21})$$

$$\begin{aligned} \mathbf{P}_{B_i|B^{i-1}}(db_i|b^{i-1}) &= \int_{\mathbb{A}^i} \mathbf{P}_{B_i|B^{i-1}, A_i}(db_i|b^{i-1}, a_i) \otimes \mathbf{P}_{A_i|A^{i-1}, B^{i-1}}(da_i|a^{i-1}, b^{i-1}) \\ &\quad \otimes \mathbf{P}_{A^{i-1}|B^{i-1}}(da^{i-1}|b^{i-1}), \quad i = 0, \dots, n. \end{aligned} \quad (\text{II.22})$$

$$\mathbf{P}_{B_0|B^{-1}}(db_0|b^{-1}) = \int_{\mathbb{A}_0} \mathbf{P}_{B_0|B^{-1}, A_0}(db_0|b^{-1}, a_0) \otimes \mathbf{P}_{A_0|B^{-1}}(da_0|b^{-1}). \quad (\text{II.23})$$

The Directed Information from $A^n \triangleq \{A_0, A_1, \dots, A_n\}$ to $B_0^n \triangleq \{B_0, B_1, \dots, B_n\}$ conditioned on B^{-1} is

¹The superscript on expectation, i.e., \mathbf{E}^g indicates the dependence of the distribution on the encoding strategies.

²If $B^{-1} = b^{-1}$ is fixed, then $\mu(\cdot) = \delta_{B^{-1}}(\cdot)$ is a dirac or delta measure concentrated at $B^{-1} = b^{-1}$.

defined by [7], [8]

$$\begin{aligned} I(A^n \rightarrow B^n) &\triangleq \sum_{i=0}^n I(A^i; B_i | B^{i-1}) = \sum_{i=0}^n I(A_i; B_i | B^{i-1}) \\ &= \sum_{i=0}^n \int \log \left(\frac{\mathbf{P}_{B_i | B^{i-1}, A_i}(\cdot | b^{i-1}, a_i)}{\mathbf{P}_{B_i | B^{i-1}}(\cdot | b^{i-1})} (b_i) \right) \mathbf{P}_{A^i, B^i}(da^i, db^i) \end{aligned} \quad (\text{II.24})$$

$$\equiv \mathbb{I}_{A^n \rightarrow B^n}^{FB}(\mathbf{P}_{A_i | A^{i-1}, B^{i-1}}, \mathbf{P}_{B_i | B^{i-1}, A_i} : i = 0, 1, \dots, n) \quad (\text{II.25})$$

where (II.24) follows from the channel definition, and the notation $\mathbb{I}_{A^n \rightarrow B^n}^{FB}(\cdot, \cdot)$ indicates that $I(A^n \rightarrow B^n)$ is a functional of the sequences of channel input and channel distributions; its dependence on the initial distribution $\mu(\cdot)$ is suppressed.

Define the information quantities

$$C_{A^n \rightarrow B^n}^{FB} \triangleq \sup_{\mathcal{P}_{[0,n]}^{FB}} I(A^n \rightarrow B^n), \quad C_{A^n \rightarrow B^n}^{FB}(\kappa) \triangleq \sup_{\mathcal{P}_{[0,n]}^{FB}(\kappa)} I(A^n \rightarrow B^n). \quad (\text{II.26})$$

Under the assumption that $\{B^{-1}, A_0, B_0, A_1, B_1, \dots\}$ is jointly ergodic or $\frac{1}{n+1} \sum_{i=0}^n \frac{\mathbf{P}_{B_i | B^{i-1}, A_i}(\cdot | b^{i-1}, a_i)}{\mathbf{P}_{B_i | B^{i-1}}(\cdot | b^{i-1})} (B_i)$ is information stable [4], [22] and $c_{0,n}(a^n, b^{n-1}) = \frac{1}{n+1} \sum_{i=0}^n \gamma_i(A_i, B^{i-1})$ is stable, then the capacity of the channel with feedback with and without transmission cost is given by

$$C_{A^\infty \rightarrow B^\infty}^{FB} \triangleq \lim_{n \rightarrow \infty} \frac{1}{n+1} C_{A^n \rightarrow B^n}^{FB}, \quad C_{A^\infty \rightarrow B^\infty}^{FB}(\kappa) \triangleq \lim_{n \rightarrow \infty} \frac{1}{n+1} C_{A^n \rightarrow B^n}^{FB}(\kappa). \quad (\text{II.27})$$

1) *Convexity Properties.*: Next, we recall the convexity properties of directed information with respect to a specific definition of channel input distributions, which is equivalent to the above definition.

Any sequence of channel input distribution $\{\mathbf{P}_{A_i | A^{i-1}, B^{i-1}} : i = 0, 1, \dots, n\} \in \mathcal{P}_{[0,n]}^{FB}$ and channel distribution $\{\mathbf{P}_{B_i | B^{i-1}, A_i} : i = 0, 1, \dots, n\}$ uniquely define the causal conditioned distributions

$$\overleftarrow{\mathbf{P}}(da^n | b^{n-1}) \triangleq \otimes_{i=0}^n \mathbf{P}_{A_i | A^{i-1}, B^{i-1}}(da_i | a^{i-1}, b^{i-1}), \quad (\text{II.28})$$

$$\overrightarrow{\mathbf{P}}(db_0^n | a^n, b^{-1}) \triangleq \otimes_{i=0}^n \mathbf{P}_{B_i | B^{i-1}, A_i}(db_i | b^{i-1}, a_i) \quad (\text{II.29})$$

and vice-versa, and these are parametrized by the initial data b^{-1} . Moreover, for a fixed $B^{-1} = b^{-1}$ we can formally define the joint distribution of $\{A_0, B_0, A_1, B_1, \dots, A_n, B_n\}$ and the joint distribution of $\{B_0, B_1, \dots, B_n\}$ conditioned on $B^{-1} = b^{-1}$ by

$$\mathbf{P}^{\overleftarrow{\mathbf{P}}}(da^n, db_0^n | b^{-1}) \triangleq (\overleftarrow{\mathbf{P}} \otimes \overrightarrow{\mathbf{P}})(da^n, db_0^n | b^{-1}), \quad (\text{II.30})$$

$$\mathbf{P}^{\overleftarrow{\mathbf{P}}}(db_0^n | b^{-1}) \triangleq \int_{\mathbb{A}^n} (\overleftarrow{\mathbf{P}} \otimes \overrightarrow{\mathbf{P}})(da^n, db_0^n | b^{-1}). \quad (\text{II.31})$$

Both distributions are parametrized by the initial data b^{-1} . Then, from [23], we have the following convexity property of directed information.

- (a) The set of conditional distributions defined by (II.28), $\overleftarrow{\mathbf{P}}_{A^n | B^{n-1}}(\cdot | b^{n-1}) \in \mathcal{M}(\mathbb{A}^n)$ is convex.
- (b) Directed information is equivalently expressed as follows.

$$\begin{aligned} I(A^n \rightarrow B^n) &= \int \log \left(\frac{\overrightarrow{\mathbf{P}}(\cdot | a^n, b^{-1})}{\overleftarrow{\mathbf{P}}(\cdot | b^{-1})} (b_0^n) \right) \mathbf{P}^{\overleftarrow{\mathbf{P}}}(da^n, db_0^n | b^{-1}) \otimes \mu(db^{-1}) \\ &\equiv \mathbb{I}_{A^n \rightarrow B^n}^{FB}(\overleftarrow{\mathbf{P}}, \overrightarrow{\mathbf{P}}). \end{aligned} \quad (\text{II.32})$$

- (c) Directed information, $\mathbb{I}_{A^n \rightarrow B^n}^{FB}(\overleftarrow{\mathbf{P}}, \overrightarrow{\mathbf{P}})$, is concave with respect to $\overleftarrow{\mathbf{P}}(\cdot | b^{n-1}) \in \mathcal{M}(\mathbb{A}^n)$ for a fixed $\overrightarrow{\mathbf{P}}(\cdot | a^n, b^{-1}) \in \mathcal{M}(\mathbb{B}_0^n)$.

Since the set of conditional distributions with or without transmission cost constraints is convex, and directed information is a concave functional, the optimization problems (II.26) are convex, and we have the following theorem.

Theorem II.1. (Convexity properties)

Assume the set $\mathcal{P}_{[0,n]}^{FB}(\kappa)$ is non-empty and the supremum of $I(A^n \rightarrow B^n)$ over the set of distributions $\mathcal{P}_{[0,n]}^{FB}(\kappa)$ is achieved (i.e., it exists). Then, the following hold.

- (a) $C_{A^n \rightarrow B^n}^{FB}(\kappa)$ is non-decreasing concave function of $\kappa \in [0, \infty]$.
- (b) An alternative characterization of $C_{A^n \rightarrow B^n}^{FB}(\kappa)$ is given by

$$C_{A^n \rightarrow B^n}^{FB}(\kappa) = \sup_{\overleftarrow{\mathbf{P}}: \frac{1}{n+1} \mathbf{E}_\mu \{c_{0,n}(a^n, b^{n-1})\} = \kappa} \mathbb{I}_{A^n \rightarrow B^n}^{FB}(\overleftarrow{\mathbf{P}}, \overrightarrow{\mathbf{P}}), \quad \text{for } \kappa \leq \kappa_{\max} \quad (\text{II.33})$$

where κ_{\max} is the smallest number belonging to $[0, \infty]$ such that $C_{A^n \rightarrow B^n}^{FB}(\kappa)$ is constant in $[\kappa_{\max}, \infty]$, and $\mathbf{E}_\mu \{ \cdot \}$ denotes expectation with respect to the joint distribution $(\overleftarrow{\mathbf{P}} \otimes \overrightarrow{\mathbf{P}}) \otimes \mu$.

Proof: Since the set $\mathcal{P}_{[0,n]}^{FB}(\kappa)$ is convex with respect to $\overleftarrow{\mathbf{P}}(\cdot|b^{n-1}) \in \mathcal{M}(\mathbb{A}^n)$, the statements follow from the convexity and non-decreasing properties [23]. \blacksquare

The above theorem states that the extremum problem of feedback capacity is a convex optimization problem, over appropriate sets of distributions.

2) *Information Structures of Optimal Channel Input Distributions.*: Consider the extremum problem $C_{A^n \rightarrow B^n}^{FB}(\kappa)$, given by (II.33). In [9], [10], it is shown that the optimal channel input distribution satisfies the following conditional independence.

$$\mathbf{P}_{A_i|A^{i-1}, B^{i-1}}(da_i|a^{i-1}, b^{i-1}) = \mathbf{P}_{A_i|B^{i-1}}(da_i|b^{i-1}) \equiv \pi_i(da_i|b^{i-1}), \quad i = 0, \dots, n. \quad (\text{II.34})$$

Moreover, in view of the information structure of the optimal channel input distribution, $C_{A^n \rightarrow B^n}^{FB}(\kappa)$ reduces to the following optimization problem.

$$C_{A^n \rightarrow B^n}^{FB}(\kappa) = \sup_{\overline{\mathcal{P}}_{[0,n]}^{FB}(\kappa)} \sum_{i=0}^n \int \log \left(\frac{\mathbf{P}_{B_i|B^{i-1}, A_i}(\cdot|b^{i-1}, a_i)}{\mathbf{P}_{B_i|B^{i-1}}^\pi(\cdot|b^{i-1})} (b_i) \right) \mathbf{P}_{A_i, B^i}^\pi(da_i, db^i) \quad (\text{II.35})$$

$$\equiv \sup_{\overline{\mathcal{P}}_{[0,n]}^{FB}(\kappa)} \mathbb{I}_{A^n \rightarrow B^n}^{FB}(\pi_i, \mathbf{P}_{B_i|B^{i-1}, A_i} : i = 0, 1, \dots, n) \quad (\text{II.36})$$

where the transmission cost constraint is defined by

$$\overline{\mathcal{P}}_{0,n}^{FB}(\kappa) \triangleq \left\{ \pi_i(da_i|b^{i-1}), i = 0, \dots, n : \frac{1}{n+1} \mathbf{E}_\mu \{c_{0,n}(A^n, B^{n-1})\} \leq \kappa \right\} \quad (\text{II.37})$$

and the induced joint and transition probability distributions are given by

$$\mathbf{P}_{A_i, B^i}^\pi(da_i, db^i) = \mathbf{P}_{B_i|B^{i-1}, A_i}(db_i|b^{i-1}, a_i) \otimes \pi_i(da_i|b^{i-1}) \otimes \mathbf{P}_{B^{i-1}}^\pi(db^{i-1}) \quad (\text{II.38})$$

$$\mathbf{P}_{B_i|B^{i-1}}^\pi(db_i|b^{i-1}) = \int_{\mathbb{A}_i} \mathbf{P}_{B_i|B^{i-1}, A_i}(db_i|b^{i-1}, a_i) \otimes \pi_i(da_i|b^{i-1}), \quad i = 0, \dots, n. \quad (\text{II.39})$$

The superscript indicates the dependence of these distributions on $\{\pi_i(da_i|b^{i-1}) : i = 0, \dots, n\}$.

The information feedback capacity rate is then given by

$$C_{A^\infty \rightarrow B^\infty}^{FB}(\kappa) \triangleq \lim_{n \rightarrow \infty} \frac{1}{n+1} \sup_{\overline{\mathcal{P}}_{[0,n]}^{FB}(\kappa)} \mathbb{I}_{A^n \rightarrow B^n}^{FB}(\pi_i, \mathbf{P}_{B_i|B^{i-1}, A_i} : i = 0, 1, \dots, n). \quad (\text{II.40})$$

C. Feedback Versus No Feedback

Here, we address the question whether feedback increases capacity via optimization problem (II.35). First, we recall the definition of channel input distributions without feedback.

Definition II.5. (Channels input distribution without feedback)

A sequence of conditional distributions defined by

$$\mathcal{P}_{[0,n]}^{noFB} \triangleq \left\{ \mathbf{P}_{A_i|A^{i-1}, B^{-1}}(da_i|a^{i-1}, b^{-1}) \equiv \pi_i^{noFB}(da_i|a^{i-1}, b^{-1}) : i = 0, \dots, n \right\}. \quad (\text{II.41})$$

The information structure of the channel input distribution without feedback is $\mathcal{I}_i^{noFB} \triangleq \{a^{i-1}, b^{-1}\}$. For time $i = 0$, the distribution is $\mathbf{P}_{A_0|A^{-1}, B^{-1}}(da_0|a^{-1}, b^{-1}) \equiv \pi_0^{noFB}(da_0|b^{-1})$, hence the information structure is $\mathcal{I}_0^{noFB} \triangleq \{a^{-1}, b^{-1}\} = \{b^{-1}\}$, which states that the channel input distribution depends only on the initial data.

Similar to the feedback case (Section II-B), the initial state of the channel, b^{-1} , is assumed to be known at the encoder. The transmission cost constraint without feedback, is defined by

$$\mathcal{P}_{[0,n]}^{noFB}(\kappa) \triangleq \left\{ \pi_i^{noFB}(da_i|a^{i-1}, b^{-1}), i = 0, \dots, n : \frac{1}{n+1} \mathbf{E}_\mu \{c_{0,n}(A^n, B^{n-1})\} \leq \kappa \right\}, \quad \kappa \in [0, \infty]. \quad (\text{II.42})$$

Moreover, the set of encoding strategies without feedback, mapping messages into channel inputs of block length $(n+1)$, are defined by

$$\begin{aligned} \mathcal{E}_{[0,n]}^{noFB}(\kappa) \triangleq \left\{ g_i^{noFB} : \mathcal{M}_n \times \mathbb{A}^{i-1} \times \mathbb{B}^{-1} \mapsto \mathbb{A}_i, \quad a_0 = g_0^{noFB}(w, b^{-1}), a_1 = g_1^{noFB}(w, b^{-1}, a_0), \dots, \right. \\ \left. a_n = g_n^{noFB}(w, b^{-1}, a^{n-1}), w \in \mathcal{M}_n : \frac{1}{n+1} \mathbf{E}^{g^{noFB}}(c_{0,n}(A^n, B^{n-1})) \leq \kappa \right\}, \quad n = 0, 1, \dots \end{aligned} \quad (\text{II.43})$$

By employing (II.42) and (II.43), a code without feedback is defined similarly to Definition II.4.

Given any channel input distribution without feedback $\{\pi_i^{noFB}(da_i|a^{i-1}, b^{-1}) : i = 0, 1, \dots, n\} \in \mathcal{P}_{[0,n]}^{noFB}(\kappa)$, a channel distribution $\{\mathbf{P}_{B_i|B^{i-1}, A_i} : i = 0, 1, \dots, n\}$, and a fixed initial distribution $\mathbf{P}_{B^{-1}}(db^{-1}) = \mu(b^{-1})$, then the induced joint distribution \mathbf{P}_{A^n, B^n} parametrized by $\mu(\cdot)$ is uniquely defined. The mutual information between from $A^n \triangleq \{A_0, A_1, \dots, A_n\}$ to $B_0^n \triangleq \{B_0, B_1, \dots, B_n\}$ conditioned on B^{-1} is defined by

$$\begin{aligned} I(A^n, B^n) &\triangleq \mathbf{E}_\mu^{\pi^{noFB}} \left\{ \log \left(\frac{\mathbf{P}_{B_0^n|A^n, B^{-1}}(\cdot|A^n, B^{-1})}{\mathbf{P}_{B_0^n|B^{-1}}(\cdot|B^{-1})} (B_0^n) \right) \right\} \\ &= \sum_{i=0}^n \int \log \left(\frac{\mathbf{P}_{B_i|B^{i-1}, A_i}(\cdot|b^{i-1}, a_i)}{\mathbf{P}_{B_i|B^{i-1}}(\cdot|b^{i-1})} (b_i) \right) \mathbf{P}_{A_i, B^i}^{\pi^{noFB}}(da^i, db^i) \\ &= \sum_{i=0}^n \int \log \left(\frac{\mathbf{P}_{B_i|B^{i-1}, A_i}(\cdot|b^{i-1}, a_i)}{\mathbf{P}_{B_i|B^{i-1}}(\cdot|b^{i-1})} (b_i) \right) \mathbf{P}_{A_i, B^i}^{\pi^{noFB}}(da_i, db^i) \\ &\equiv \mathbb{I}_{A^n \rightarrow B^n}^{noFB}(\pi_i^{noFB}, \mathbf{P}_{B_i|B^{i-1}, A_i} : i = 0, 1, \dots, n) \end{aligned} \quad (\text{II.44})$$

where the joint distribution and transition probability distribution are induced by $\{\pi_i^{noFB}(da_i|a^{i-1}, b^{-1}) : i = 0, \dots, n\} \in \mathcal{P}_{[0,n]}^{noFB}$ as follows.

$$\mathbf{P}_{A_i, B^i}^{\pi^{noFB}}(da_i, db^i) = \mathbf{P}_{B_i|B^{i-1}, A_i}(db_i|b^{i-1}, a_i) \otimes \mathbf{P}_{A_i|B^{i-1}}^{\pi^{noFB}}(da_i|b^{i-1}) \otimes \mathbf{P}_{B^{i-1}}^{\pi^{noFB}}(db^{i-1}) \quad (\text{II.45})$$

$$\mathbf{P}_{B_i|B^{i-1}}^{\pi^{noFB}}(db_i|b^{i-1}) = \int_{\mathbb{A}_i} \mathbf{P}_{B_i|B^{i-1}, A_i}(db_i|b^{i-1}, a_i) \otimes \mathbf{P}_{A_i|B^{i-1}}^{\pi^{noFB}}(da_i|b^{i-1}), \quad i = 0, \dots, n \quad (\text{II.46})$$

$$\mathbf{P}_{A_i|B^{i-1}}^{\pi^{noFB}}(da_i|b^{i-1}) = \int_{\mathbb{A}^{i-1}} \pi_i^{noFB}(da_i|a^{i-1}, b^{-1}) \otimes \mathbf{P}_{A^{i-1}|B^{i-1}}^{\pi^{noFB}}(da^{i-1}|b^{i-1}) \quad (\text{II.47})$$

$$\mathbf{P}_{A^{i-1}|B^{i-1}}^{\pi^{noFB}}(da^{i-1}|b^{i-1}) = \otimes_{j=0}^{i-1} \frac{\mathbf{P}_{B_j|B^{j-1}, A_j}(db_j|b^{j-1}, a_j) \otimes \pi_j^{noFB}(da_j|a^{j-1}, b^{-1})}{\int_{\mathbb{A}_j} \mathbf{P}_{B_j|B^{j-1}, A_j}(db_j|b^{j-1}, a_j) \otimes \pi_j^{noFB}(da_j|a^{j-1}, b^{-1})}. \quad (\text{II.48})$$

The superscript in the above distributions are important to distinguish that these are generated by the channel and channel input distributions without feedback, while the functional in (II.44) is fundamentally different from the one in (II.36). Clearly, compared to the channel with feedback in which the corresponding distributions are (II.38) and (II.39), and they are induced by $\{\pi_i(da_i|b^{i-1}) : i = 0, \dots, n\} \in$

$\overline{\mathcal{P}}_{0,n}^{FB}(\kappa)$, when the channel is used without feedback, the distributions (II.45) and (II.46) are induced by $\{\pi_i^{noFB}(da_i|a^{i-1}, b^{-1}) : i = 0, \dots, n\} \in \mathcal{P}_{[0,n]}^{noFB}(\kappa)$.

Define the information quantity

$$C_{A^n;B^n}^{noFB}(\kappa) = \sup_{\mathcal{P}_{[0,n]}^{noFB}(\kappa)} \sum_{i=0}^n \int \log \left(\frac{\mathbf{P}_{B_i|B^{i-1},A_i}(\cdot|b^{i-1}, a_i)}{\mathbf{P}_{B_i|B^{i-1}}(\cdot|b^{i-1})}(b_i) \right) \mathbf{P}_{A_i,B_i}^{\pi_i^{noFB}}(da_i, db_i) \quad (\text{II.49})$$

$$\equiv \sup_{\mathcal{P}_{[0,n]}^{noFB}(\kappa)} \mathbb{I}_{A^n \rightarrow B^n}^{noFB}(\pi_i^{noFB}, \mathbf{P}_{B_i|B^{i-1},A_i} : i = 0, 1, \dots, n). \quad (\text{II.50})$$

Then the information capacity without feedback subject to a transmission cost constraint is defined by

$$C_{A^\infty;B^\infty}^{noFB}(\kappa) \triangleq \lim_{n \rightarrow \infty} \frac{1}{n+1} \sup_{\mathcal{P}_{[0,n]}^{noFB}(\kappa)} \mathbb{I}_{A^n \rightarrow B^n}^{noFB}(\pi_i^{noFB}, \mathbf{P}_{B_i|B^{i-1},A_i} : i = 0, 1, \dots, n). \quad (\text{II.51})$$

Next, we note the following. Let $\{\pi_i^*(da_i|b^{i-1}) : i = 0, \dots, n\} \in \overline{\mathcal{P}}_{[0,n]}^{FB}(\kappa)$ denote the maximizing distribution in $C_{A^n \rightarrow B^n}^{FB}(\kappa)$ defined by (II.35). Suppose there exists a sequence of channel input distributions without feedback $\{\mathbf{P}^*(da_i|\mathcal{I}_i^{noFB}) \equiv \pi_i^{*,noFB}(da_i|\mathcal{I}_i^{noFB}) : \mathcal{I}_i^{noFB} \subseteq \{b^{-1}, a_0, \dots, a_{i-1}\}, i = 0, \dots, n\} \in \mathcal{P}_{0,n}^{noFB}(\kappa)$ which induces the maximizing channel input distribution with feedback $\{\pi_i^*(da_i|b^{i-1}) : i = 0, 1, \dots, n\}$. That is, $\mathbf{P}_{A_i|B^{i-1}}^{noFB}(da_i|b^{i-1})$ given by (II.47) is equal to $\pi_i^*(da_i|b^{i-1})$, $\forall i = 0, 1, \dots, n$. Then, it is clear that this sequence also induces the optimal joint distribution and conditional distribution defined by (II.38), (II.39), and consequently $C_{A^n \rightarrow B^n}^{FB}(\kappa)$ and $C_{A^\infty \rightarrow B^\infty}^{FB}(\kappa)$ are achieved without using feedback.

In the following theorem, we prove that this condition is not only sufficient but also necessary for any channel input distribution without feedback to achieve the finite time feedback information capacity, $C_{A^n \rightarrow B^n}^{FB}(\kappa)$.

Theorem II.2. (Necessary and sufficient conditions for $C_{A^n \rightarrow B^n}^{FB}(\kappa) = C_{A^n;B^n}^{noFB}(\kappa)$)

Consider channel (II.15) and let $\{\pi_i^*(da_i|b^{i-1}) : i = 0, \dots, n\} \in \overline{\mathcal{P}}_{[0,n]}^{FB}(\kappa)$ denote the maximizing distribution in $C_{A^n \rightarrow B^n}^{FB}(\kappa)$ defined by (II.35), and let $\{\mathbf{P}_{A_i,B_i}^{\pi_i^*}(da_i, db_i), \mathbf{P}_{B_i|B^{i-1}}^{\pi_i^*}(db_i|b^{i-1}) : i = 0, \dots, n\}$ denote the corresponding joint and transition distributions as defined by (II.38), (II.39).

Then

$$C_{A^n \rightarrow B^n}^{FB}(\kappa) = C_{A^n;B^n}^{noFB}(\kappa) \quad (\text{II.52})$$

if and only if there exists a sequence of channel input distributions

$$\{\mathbf{P}^*(da_i|\mathcal{I}_i^{noFB}) \equiv \pi_i^{noFB,*}(da_i|\mathcal{I}_i^{noFB}) : \mathcal{I}_i^{noFB} \subseteq \{b^{-1}, a_0, \dots, a_{i-1}\}, i = 0, \dots, n\} \in \mathcal{P}_{0,n}^{noFB}(\kappa)$$

which induces the maximizing channel input distribution with feedback $\{\pi_i^*(da_i|b^{i-1}) : i = 0, 1, \dots, n\}$.

Proof: In general, the inequality $C_{A^n \rightarrow B^n}^{FB}(\kappa) \geq C_{A^n;B^n}^{noFB}(\kappa)$ holds. Moreover, by Section II-B the distributions $\{\mathbf{P}_{A_i,B_i}^{\pi_i^*}(da_i, db_i), \mathbf{P}_{B_i|B^{i-1}}^{\pi_i^*}(db_i|b^{i-1}) : i = 0, \dots, n\}$ are induced by the channel, which is fixed, and the optimal conditional distribution $\{\pi_i^*(da_i|b^{i-1}) : i = 0, \dots, n\} \in \overline{\mathcal{P}}_{[0,n]}^{FB}(\kappa)$. Then, equality holds if and only if there exists a distribution without feedback $\{\pi_i^{noFB,*}(da_i|\mathcal{I}_i^{noFB}) : i = 0, \dots, n\} \in \mathcal{P}_{0,n}^{noFB}(\kappa)$ which induces $\{\pi_i^*(da_i|b^{i-1}) : i = 0, \dots, n\} \in \overline{\mathcal{P}}_{[0,n]}^{FB}(\kappa)$. This follows from the fact that the distributions $\{\mathbf{P}_{A_i,B_i}^{\pi_i^*}(da_i, db_i), \mathbf{P}_{B_i|B^{i-1}}^{\pi_i^*}(db_i|b^{i-1}) : i = 0, \dots, n\}$ are induced by the feedback distribution, $\{\pi_i^*(da_i|b^{i-1}) : i = 0, \dots, n\}$ and the channel distribution. This completes the proof. ■

Theorem II.2 provides a sufficient condition for feedback not to increase capacity, i.e. $C_{A^\infty \rightarrow B^\infty}^{FB}(\kappa) = C_{A^\infty;B^\infty}^{noFB}(\kappa)$, since if (II.52) holds, then $\lim_{n \rightarrow \infty} \frac{1}{n+1} C_{A^n \rightarrow B^n}^{FB}(\kappa) = \lim_{n \rightarrow \infty} \frac{1}{n+1} C_{A^n;B^n}^{noFB}(\kappa)$. In Section IV-B we demonstrate an application of Theorem II.2 to a specific channel with memory, where we show that an input distribution without feedback induces $\{\pi_i^*(da_i|b^{i-1}) : i = 0, \dots, n\}$, hence feedback does not increase capacity.

III. DYNAMIC PROGRAMMING AND NECESSARY SUFFICIENT CONDITIONS FOR NON-NESTED OPTIMIZATION

In this section we employ the structural properties of capacity achieving channel input distributions with feedback to derive dynamic programming recursions and necessary and sufficient conditions for the single letter characterization (I.6), to hold. Specifically, we provide the following results for the UMCO channel.

- (a) Necessary and sufficient conditions to determine when dynamic programming recursions, which are nested optimization problems, reduce to non-nested optimization problems.
- (b) Repeat (a) for the per unit time infinite horizon.
- (c) Upper bounds on the probability of maximum likelihood decoding.

The time-varying UMCO channel is defined by

$$\mathbf{P}_i(b_i|b^{i-1}, a^i) \triangleq \mathbf{P}_i(b_i|b_{i-1}, a_i), \quad i = 0, \dots, n \quad (\text{III.53})$$

and the transmission cost constraint is defined by

$$\frac{1}{n+1} \mathbf{E} \left\{ \sum_{i=0}^n \gamma_i^{UM}(A_i, B_{i-1}) \right\} \leq \kappa \quad (\text{III.54})$$

where $\gamma_i^{UM} : \mathbb{A}_i \times \mathbb{B}_{i-1} \mapsto [0, \infty)$. At $i = 0$ the conditional distribution depends on $\{b_{-1}, a_0\}$, where $b_{-1} \in \mathbb{B}_{-1}$ is the initial data which are either known to the encoder and the decoder or, $b_{-1} = \{\emptyset\}$. For simplicity, of presentation and technical assumptions needed, we consider a channel model with transmission cost function, defined on finite alphabet spaces. However, all main results extend to abstract alphabet spaces and channel distributions, which depend on finite memory on past channel output. Moreover, our analysis and the corresponding theorems can be extended to channels with finite memory on the previous channel outputs by exploiting the structural form of the capacity achieving distributions given in [9], [10].

For the above model, it is shown in [9], [10] that maximizing directed information, $I(A^n \rightarrow B^n)$, over $\mathcal{P}_{[0,n]}^{FB}$ or $\mathcal{P}_{[0,n]}^{FB}(\kappa)$ occurs in the subset of conditional distributions that satisfy the following conditional independence.

$$\mathbf{P}_i(a_i|a^{i-1}, b^{i-1}) = \mathbf{P}_i(a_i|b_{i-1}) \equiv \pi_i(a_i|b_{i-1}), \quad i = 0, 1, \dots, n. \quad (\text{III.55})$$

Consequently, we have the following Markovian properties.

$$\mathbf{P}_i(a_i, b_i|a^{i-1}, b^{i-1}) = \mathbf{P}_i^\pi(a_i, b_i|a_{i-1}, b_{i-1}), \quad i = 0, 1, \dots, n, \quad (\text{III.56})$$

$$\mathbf{P}_i(b_i|b^{i-1}) = \mathbf{P}_i^\pi(b_i|b_{i-1}), \quad i = 0, 1, \dots, n, \quad (\text{III.57})$$

$$\mathbf{P}_i^\pi(b_i|b_{i-1}) = \sum_{a_i \in \mathbb{A}_i} \mathbf{P}_i(b_i|b_{i-1}, a_i) \pi_i(a_i|b_{i-1}), \quad i = 0, 1, \dots, n. \quad (\text{III.58})$$

where the superscript indicates the dependence on the channel input distribution (III.55). In view of these Markov properties, the characterization of the FTFI capacity (i.e., (II.26)) is given by³

$$C_{A^n \rightarrow B^n}^{FB, UMCO} \triangleq \sup_{\substack{\circ \\ \mathcal{P}_{[0,n]}^{FB}}} \mathbf{E}_\mu^\pi \left\{ \sum_{i=0}^n \log \left(\frac{\mathbf{P}_i(B_i|B_{i-1}, A_i)}{\mathbf{P}_i^\pi(B_i|B_{i-1})} \right) \right\} \quad (\text{III.59})$$

$$= \sup_{\substack{\circ \\ \mathcal{P}_{[0,n]}^{FB}}} \sum_{i=0}^n I(A_i; B_i|B_{i-1}) \quad (\text{III.60})$$

where

$$\substack{\circ \\ \mathcal{P}_{[0,n]}^{FB}} \triangleq \{ \pi_i(a_i|b_{i-1}) : i = 0, 1, \dots, n \} \subset \mathcal{P}_{[0,n]}^{FB}. \quad (\text{III.61})$$

³When clear from the context, the subscript notation of the distributions is omitted, i.e., $\mathbf{P}_{B_i|B_{i-1}}^\pi(b_i|b_{i-1}) \equiv \mathbf{P}_i^\pi(b_i|b_{i-1})$.

Similarly, for conditional distributions with transmission cost the characterization of FTFI capacity is given by

$$C_{A^n \rightarrow B^n}^{FB,UMCO}(\kappa) \triangleq \sup_{\overset{\circ}{\mathcal{P}}_{[0,n]}^{FB}(\kappa)} \mathbf{E}_{\mu}^{\pi} \left\{ \sum_{i=0}^n \log \left(\frac{\mathbf{P}_i(B_i|B_{i-1}, A_i)}{\mathbf{P}_i^{\pi}(B_i|B_{i-1})} \right) \right\} \quad (\text{III.62})$$

$$= \sup_{\overset{\circ}{\mathcal{P}}_{[0,n]}^{FB}(\kappa)} \sum_{i=0}^n I(A_i; B_i | B_{i-1}) \quad (\text{III.63})$$

where

$$\overset{\circ}{\mathcal{P}}_{[0,n]}^{FB}(\kappa) \triangleq \left\{ \pi_i(a_i|b_{i-1}), i = 0, 1, \dots, n : \frac{1}{n+1} \sum_{i=0}^n \mathbf{E}_{\mu}^{\pi} \left(\gamma_i^{UM}(A_i, B_{i-1}) \right) \leq \kappa \right\}. \quad (\text{III.64})$$

Since the joint process $\{B_{-1}, A_0, B_0, \dots, A_n, B_n\}$ and channel output process $\{B_{-1}, B_0, \dots, B_n\}$ are Markov, we explore the connection of the above optimization problems to Markov Decision theory, to derive the results listed in (a)-(c). We do this in the next sections.

A. Necessary and Sufficient Conditions via Dynamic Programming: The Finite Horizon case

To derive the necessary and sufficient conditions for any channel input distribution to maximize directed information, i.e., item (a), we first apply dynamic programming on a finite horizon.

1) *Without Transmission Cost Constraint:* The dynamic programming recursion for $C_{A^n \rightarrow B^n}^{FB,UMCO}$ is obtained as follows. Let $V_t(b_{t-1})$ represent the value function, that is, the maximum expected total cost on the future time horizon $\{t, t+1, \dots, n\}$ given output $B_{t-1} = b_{t-1}$ at time $t-1$, defined by

$$V_t(b_{t-1}) = \sup_{\pi_i(a_i|b_{i-1}): i=t, t+1, \dots, n} \mathbf{E}^{\pi} \left\{ \sum_{i=t}^n \log \left(\frac{\mathbf{P}_i(B_i|B_{i-1}, A_i)}{\mathbf{P}_i^{\pi}(B_i|B_{i-1})} \right) | B_{t-1} = b_{t-1} \right\} \quad (\text{III.65})$$

where the transition probability of the channel output process is

$$\mathbf{P}_t^{\pi}(b_t|b_{t-1}) = \sum_{a_t \in \mathbb{A}_t} \mathbf{P}_t(b_t|b_{t-1}, a_t) \pi_t(a_t|b_{t-1}). \quad (\text{III.66})$$

Then (III.65) satisfies the following dynamic programming recursions.

$$V_n(b_{n-1}) = \sup_{\pi_n(a_n|b_{n-1})} \sum_{(a_n, b_n) \in \mathbb{A}_n \times \mathbb{B}_n} \log \left(\frac{\mathbf{P}_n(b_n|b_{n-1}, a_n)}{\mathbf{P}_n^{\pi}(b_n|b_{n-1})} \right) \mathbf{P}_n(b_n|b_{n-1}, a_n) \pi_n(a_n|b_{n-1}), \quad (\text{III.67})$$

$$V_t(b_{t-1}) = \sup_{\pi_t(a_t|b_{t-1})} \left\{ \sum_{a_t \in \mathbb{A}_t} \left[\sum_{b_t \in \mathbb{B}_t} \log \left(\frac{\mathbf{P}_t(b_t|b_{t-1}, a_t)}{\mathbf{P}_t^{\pi}(b_t|b_{t-1})} \right) \mathbf{P}_t(b_t|b_{t-1}, a_t) \right. \right. \\ \left. \left. + \sum_{b_t \in \mathbb{B}_t} V_{t+1}(b_t) \mathbf{P}_t(b_t|b_{t-1}, a_t) \right] \pi_t(a_t|b_{t-1}) \right\}, \quad t = 0, 1, \dots, n-1. \quad (\text{III.68})$$

For a fixed initial distribution $\mathbf{P}_{B_{-1}}(b_{-1}) = \mu(b_{-1})$ we have

$$C_{A^n \rightarrow B^n}^{FB,UMCO} = \sum_{b_{-1} \in \mathbb{B}_{-1}} V_0(b_{-1}) \mu(b_{-1}). \quad (\text{III.69})$$

Clearly, by using the properties of relative entropy, we can show that the right hand side of the dynamic programming recursion, (III.67), is a concave function of the input distribution $\pi_n(a_n|b_{n-1})$. Similarly at each step of the recursion, the right hand side of the dynamic programming recursion, (III.68), is a concave function of the input distribution $\pi_t(a_t|b_{t-1})$, since the future channel input distributions, $\{\pi_{t+1}(a_{t+1}|b_t), \dots, \pi_n(a_n|b_{n-1})\}$ are fixed to their optimal strategies. Utilizing this observation we have the following necessary and sufficient conditions for any channel input distribution to maximize the right hand side of the dynamic programming recursions (III.67) and (III.68).

Theorem III.1. (Necessary and sufficient conditions)

The necessary and sufficient conditions for any input distribution $\{\pi_t(a_t|b_{t-1}) : t = 0, 1, \dots, n\}$ to achieve the supremum of the dynamic programming recursions (III.67) and (III.68) are the following. For each $b_{n-1} \in \mathbb{B}_{n-1}$, there exist $V_n(b_{n-1})$ such that

$$V_n(b_{n-1}) = \sum_{b_n \in \mathbb{B}_n} \log \left(\frac{\mathbf{P}_n(b_n|a_n, b_{n-1})}{\mathbf{P}_n^\pi(b_n|b_{n-1})} \right) \mathbf{P}_n(b_n|a_n, b_{n-1}), \quad \forall a_n \in \mathbb{A}_n \text{ if } \pi_n(a_n|b_{n-1}) \neq 0, \quad (\text{III.70})$$

$$V_n(b_{n-1}) \leq \sum_{b_n \in \mathbb{B}_n} \log \left(\frac{\mathbf{P}_n(b_n|a_n, b_{n-1})}{\mathbf{P}_n^\pi(b_n|b_{n-1})} \right) \mathbf{P}_n(b_n|a_n, b_{n-1}), \quad \forall a_n \in \mathbb{A}_n \text{ if } \pi_n(a_n|b_{n-1}) = 0 \quad (\text{III.71})$$

and for each $t = n-1, n-2, \dots, 1, 0$ there exist $V_t(b_{t-1})$ such that

$$V_t(b_{t-1}) = \sum_{b_t \in \mathbb{B}_t} \left\{ \log \left(\frac{\mathbf{P}_t(b_t|a_t, b_{t-1})}{\mathbf{P}_t^\pi(b_t|b_{t-1})} \right) + V_{t+1}(b_t) \right\} \mathbf{P}_t(b_t|a_t, b_{t-1}), \quad \forall a_t \in \mathbb{A}_t, \text{ if } \pi_t(a_t|b_{t-1}) \neq 0, \quad (\text{III.72})$$

$$V_t(b_{t-1}) \leq \sum_{b_t \in \mathbb{B}_t} \left\{ \log \left(\frac{\mathbf{P}_t(b_t|a_t, b_{t-1})}{\mathbf{P}_t^\pi(b_t|b_{t-1})} \right) + V_{t+1}(b_t) \right\} \mathbf{P}_t(b_t|a_t, b_{t-1}), \quad \forall a_t \in \mathbb{A}_t, \text{ if } \pi_t(a_t|b_{t-1}) = 0. \quad (\text{III.73})$$

Moreover, $\{V_t(b_{t-1}) : (t, b_{t-1}) \in \{0, \dots, n\} \times \mathbb{B}_{t-1}\}$ is the value function defined by (III.65).

Proof: The derivation is given in [24]. ■

Before we proceed further, in the next remark, we relate Theorem III.1 to the necessary and sufficient conditions of DMCs derived in [25].

Remark III.1. (Relation to necessary and sufficient conditions of DMCs)

(a) Suppose the channel is a time-varying DMC, i.e.,

$$\mathbf{P}_t(b_t|b_{t-1}, a_t) = \mathbf{P}_t(b_t|a_t), \quad t = 0, \dots, n. \quad (\text{III.74})$$

Since the optimal distribution of DMCs, which maximizes the directed information $I(A^n \rightarrow B^n)$ is memoryless, i.e., $\mathbf{P}_{A_t|A^{t-1}, B^{t-1}}(a_t|a^{t-1}, b^{t-1}) = \mathbf{P}_{A_t}(a_t) \equiv \pi_t(a_t), t = 0, \dots, n$, then (III.66) reduces to $\mathbf{P}_t^\pi(b_t) = \int_{\mathbb{A}_t} \mathbf{P}_t(b_t|a_t) \pi_t(a_t), t = 0, \dots, n$. By replacing in (III.70) and (III.71), the following quantities

$$\mathbf{P}_t(b_t|b_{t-1}, a_t) \mapsto \mathbf{P}_t(b_t|a_t), \quad \mathbf{P}_t^\pi(b_t|b_{t-1}) \mapsto \mathbf{P}_t^\pi(b_t) = \int_{\mathbb{A}_t} \mathbf{P}_t(b_t|a_t) \pi_t(a_t), t = 0, \dots, n \quad (\text{III.75})$$

we obtain

$$V_n(b_{n-1}) \equiv \bar{V}_n = \sum_{b_n} \log \left(\frac{\mathbf{P}_n(b_n|a_n)}{\mathbf{P}_n^\pi(b_n)} \right) \mathbf{P}_n(b_n|a_n), \quad \forall a_n \in \mathbb{A}_n \text{ if } \pi_n(a_n) \neq 0, \quad (\text{III.76})$$

$$V_n(b_{n-1}) \equiv \bar{V}_n \leq \sum_{b_n} \log \left(\frac{\mathbf{P}_n(b_n|a_n)}{\mathbf{P}_n^\pi(b_n)} \right) \mathbf{P}_n(b_n|a_n), \quad \forall a_n \in \mathbb{A}_n \text{ if } \pi_n(a_n) = 0 \quad (\text{III.77})$$

where $V_n(b_{n-1}) = \bar{V}_n$ is a constant number, independent of b_{n-1} . Moreover, for each t , from (III.72) and (III.73) we obtain

$$V_t(b_{t-1}) \equiv \bar{V}_t = \sum_{b_t} \log \left(\frac{\mathbf{P}_t(b_t|a_t)}{\mathbf{P}_t^\pi(b_t)} \right) \mathbf{P}_t(b_t|a_t) + \bar{V}_{t+1}, \quad \text{if } \forall a_t \in \mathbb{A}_t, \quad \pi_t(a_t) \neq 0, \quad (\text{III.78})$$

$$V_t(b_{t-1}) \equiv \bar{V}_t \leq \sum_{b_t} \log \left(\frac{\mathbf{P}_t(b_t|a_t)}{\mathbf{P}_t^\pi(b_t)} \right) \mathbf{P}_t(b_t|a_t) + \bar{V}_{t+1}, \quad \text{if } \forall a_t \in \mathbb{A}_t, \quad \pi_t(a_t) = 0 \quad (\text{III.79})$$

where $V_t(b_{t-1}) \equiv \bar{V}_t$ is a constant number independent of b_{t-1} , for $t = n-1, n-2, \dots, 1, 0$. Consequently,

by evaluating $V_t(b_{t-1}) = \bar{V}_t$ at $t = 0$, we obtain the following identities.

$$\bar{V}_0 = \max_{\pi_t(a_t): t=0, \dots, n} \mathbf{E}^\pi \left\{ \sum_{t=0}^n \log \left(\frac{\mathbf{P}_t(b_t|a_t)}{\mathbf{P}_t^\pi(b_t)} \right) \right\} = \sum_{t=0}^n \max_{\pi_t(a_t)} \mathbf{E}^\pi \left\{ \log \left(\frac{\mathbf{P}_t(b_t|a_t)}{\mathbf{P}_t^\pi(b_t)} \right) \right\}. \quad (\text{III.80})$$

As expected, (III.80) shows that under (III.74), the sequence of nested optimization problems reduces to a sequence of non-nested optimization problems.

(b) Suppose the channel is time-invariant (homogeneous) DMC. In this case, $\mathbf{P}_t(b_t|b_{t-1}, a_t) = \mathbf{P}(b_t|b_{t-1}, a_t)$, $t = 0, \dots, n$, and the equations in (a) reduce to the single set of necessary and sufficient conditions obtained in [25], that is, letting $\bar{V} = C \triangleq \max_{\mathbf{P}_A} I(A; B)$, then

$$\bar{V} = \sum_b \log \left(\frac{\mathbf{P}(b|a)}{\mathbf{P}^\pi(b)} \right) \mathbf{P}(b|a), \quad \forall a \in \mathbb{A} \text{ if } \pi(a) \neq 0, \quad (\text{III.81})$$

$$\bar{V} \leq \sum_b \log \left(\frac{\mathbf{P}(b|a)}{\mathbf{P}^\pi(b)} \right) \mathbf{P}(b|a), \quad \forall a \in \mathbb{A} \text{ if } \pi(a) = 0. \quad (\text{III.82})$$

In view of Remark III.1, next we identify necessary and sufficient conditions for any optimal channel input conditional distribution, which is a solution of the dynamic programming recursions to be time-invariant, i.e., item (b), and to exhibit a non-nested property reminiscent to that of DMCs, i.e., item (c).

We derived such conditions based on the following definition.

Definition III.1. (Non-nested optimization)

Given a channel distribution $\{\mathbf{P}_t(b_t|b_{t-1}, a_t) : t = 0, \dots, n\}$, the optimization problem $C_{A^n \rightarrow B^n}^{FB, UMCO}$ defined by (III.69) is called

(a) non-nested if and only if the value function (III.65) satisfies the following non-nested identity.

$$V_t(b_{t-1}) = \sum_{i=t}^n \sup_{\pi_i(a_i|b_{i-1})} \mathbf{E}^\pi \left\{ \log \left(\frac{\mathbf{P}_i(B_i|B_{i-1}, A_i)}{\mathbf{P}_i^\pi(B_i|B_{i-1})} \right) \middle| B_{i-1} = b_{i-1} \right\} \quad (\text{III.83})$$

for all $(t, b_{t-1}) \in \{0, 1, \dots, n\} \times \mathbb{B}_{t-1}$;

(b) non-nested and time-invariant if and only if the value function satisfies the following identity.

$$V_t(b_{t-1}) = (n - t + 1) \sup_{\pi_t(a_t|b_{t-1})} \mathbf{E}^\pi \left\{ \log \left(\frac{\mathbf{P}_t(B_t|B_{t-1}, A_t)}{\mathbf{P}_t^\pi(B_t|B_{t-1})} \right) \middle| B_{t-1} = b_{t-1} \right\} \quad (\text{III.84})$$

for all $(t, b_{t-1}) \in \{0, 1, \dots, n\} \times \mathbb{B}_{t-1}$.

Clearly, if we can identify conditions so that the optimization problem defined by (III.69) is non-nested, then by evaluating the value function (III.83) at time $t = 0$ we obtain the analogue of (III.80), for channels with memory. This means that the optimal channel input distribution at each time instant is obtained by maximizing $I(A_i; B_i | B_{i-1} = b_{i-1})$ over $\pi(a_i | b_{i-1})$, for which b_{i-1} is fixed. Moreover, if the optimization problem is non-nested and time invariant, then by evaluating (III.84) at $t = 0$, we obtain

$$C_{A^n \rightarrow B^n}^{FB, UMCO} = (n + 1) \sum_{b_{-1} \in \mathbb{B}_{-1}} I(A_0; B_0 | B_{-1} = b_{-1}). \quad (\text{III.85})$$

Next, we state the main theorem, which generalizes the non-nested and time-invariant properties of memoryless channels given in Remark III.1, to channels with memory.

Theorem III.2. (Necessary and sufficient conditions for non-nested optimization)

(a) Consider any channel distribution $\{\mathbf{P}_i(b_i|b_{i-1}, a_i) : i = 0, \dots, n\}$.

The optimization problem $C_{A^n \rightarrow B^n}^{FB, UMCO}$ defined by (III.69) is non-nested and the value function is charac-

terized by (III.83) if and only if

there exists constants $\{\bar{V}_t : t = 0, \dots, n\}$ such that $V_t(b_{t-1}) = \bar{V}_t$, $\forall (t, b_{t-1}) \in \{0, 1, \dots, n\} \times \mathbb{B}_{t-1}$ which satisfy (III.70)-(III.73). (III.86)

(b) Consider any time-invariant channel distribution $\{\mathbf{P}(b_i|b_{i-1}, a_i) : i = 0, \dots, n\}$.

The optimization problem $C_{A^n \rightarrow B^n}^{FB, UMC}$ defined by (III.69) is non-nested and time-invariant and the value function is characterized by

$$V_t(b_{t-1}) = \bar{V}_t \triangleq (n-t+1) \sup_{\pi^{TI}(a_i|b_{i-1})} \mathbf{E}^\pi \left\{ \log \left(\frac{\mathbf{P}(B_i|B_{i-1}, A_i)}{\mathbf{P}^{\pi^{TI}}(B_i|B_{i-1})} \right) \middle| B_{i-1} = b_{i-1} \right\},$$

$$\forall (t, b_{t-1}) \in \{0, 1, \dots, n\} \times \mathbb{B}_{t-1} \quad (III.87)$$

where $\{\pi_i(a_i|b_{i-1}) = \pi^{TI}(a_i|b_{i-1}) : i = 0, \dots, n\}$ and $\{\mathbf{P}_i^\pi(b_i|b_{i-1}) = \mathbf{P}^{\pi^{TI}}(b_i|b_{i-1}) : i = 0, \dots, n\}$ are time-invariant, if and only if

there exists a constant \bar{V}_n such that $V_n(b_{n-1}) = \bar{V}_n$, $\forall b_{n-1} \in \mathbb{B}_{n-1}$ which satisfies (III.70), (III.71). (III.88)

Proof: (a) Suppose (III.86) holds. Then by Theorem III.1, for any t , the optimal strategy $\pi_t(a_t|b_{t-1})$ is not affected by the future strategies $\{\pi_i(a_i|b_{i-1}) : i = t+1, t+2, \dots, n\}$ for all $t = 0, 1, \dots, n-1$. Hence, the optimization problem $C_{A^n \rightarrow B^n}^{FB, UMC}$ is non-nested. Conversely, if (III.83) holds, since its left hand side is the value function defined by (III.65), then necessarily for each t , the value function is a constant, i.e., $\{V_i(b_{i-1}) = \bar{V}_i : i = t+1, \dots, n\}$, for $t = 0, 1, \dots, n-1$. In view of Theorem III.1, then (III.86) holds.

(b) This is degenerate case of part (a). Suppose (III.88) holds and consider the necessary and sufficient conditions given in Theorem III.1 at time $t = n-1$. Since $V_n(b_{n-1}) = \bar{V}_n, \forall b_{n-1}$, then by (III.72) (and similarly for (III.73)) we have $V_{n-1}(b_{n-2}) = \{\cdot\} + \bar{V}_n, \forall a_{n-1}, \pi_{n-1}(a_{n-1}|b_{n-2})$, where the term $\{\cdot\}$ is the first right hand side term in (III.72). Since the channel is time-invariant, subtracting the term \bar{V}_n from both sides of the equation (III.72) (i.e., corresponding to $t = n-1$), then the resulting equations are precisely (III.70) (and similarly for (III.71)). Thus, (III.84) holds for $t = n-1$. To complete the derivation, we use induction, that is, we assume validity of (III.84) for $t \in \{n, n-1, \dots, i+1\}$ and we show it also holds for $t = i$. This is similar to the case $t = n-1$ hence it is omitted. Conversely, if (III.84) holds, then using the time-invariant property of the channel, then necessarily (III.88) holds (as in part (a)). ■

Clearly, the above theorem is a generalization of Remark III.1 to channels with memory. In Section IV we present one example. However, additional ones can be identified by invoking the necessary and sufficient conditions of Theorem III.1 and Theorem III.2.

2) *With Transmission Cost Constraints.*: All statements of the previous section generalize to $C_{A^n \rightarrow B^n}^{FB, UMC}(\kappa)$ defined by (III.63), where the transmission cost constraint is given by (III.64). In view of the convexity of the optimization problem, and existence of an interior point of the constraint set $\mathcal{P}_{[0,n]}^{FB}(\kappa)$ (i.e., Slater's condition), by Lagrange duality theorem [26], then the constraint and unconstrained problems are equivalent, that is,

$$C_{A^n \rightarrow B^n}^{FB, UMC}(\kappa) = \inf_{s \geq 0} \sup_{\pi_i(a_i|b_{i-1}) : i=0,1,\dots,n} \mathbf{E}_\mu^\pi \left\{ \sum_{i=0}^n \left[\log \left(\frac{\mathbf{P}_i^\pi(b_i|b_{i-1}, a_i)}{\mathbf{P}_i^\pi(b_i|b_{i-1})} \right) - s(\gamma^{UM}(A_i, B_{i-1}) - (n+1)\kappa) \right] \right\} \quad (III.89)$$

where $s \in [0, \infty)$ is the Lagrange multiplier associated with the constraint.

The dynamic programming recursions are obtained as follows. Let $V_t^s(b_{t-1})$ represent value function on the future time horizon $\{t, t+1, \dots, n\}$ given output $B_{i-1} = b_{i-1}$ at time $t-1$, defined by

$$V_t^s(b_{t-1}) = \sup_{\pi_i(a_i|b_{i-1}) : i=t,t+1,\dots,n} \mathbf{E}^\pi \left\{ \sum_{i=t}^n \left[\log \left(\frac{\mathbf{P}_i(B_i|B_{i-1}, A_i)}{\mathbf{P}_i^\pi(B_i|B_{i-1})} \right) - s\gamma^{UM}(A_i, B_{i-1}) \right] \middle| B_{t-1} = b_{t-1} \right\}. \quad (III.90)$$

The corresponding dynamic programming recursions are the following.

$$V_n^s(b_{n-1}) = \sup_{\pi_n(a_n|b_{n-1})} \left\{ \sum_{a_n \in \mathbb{A}_n} \left[\sum_{b_n \in \mathbb{B}_n} \log \left(\frac{\mathbf{P}_n(b_n|b_{n-1}, a_n)}{\mathbf{P}_n^\pi(b_n|b_{n-1})} \right) \mathbf{P}_n(b_n|b_{n-1}, a_n) - s\gamma_n^{UM}(a_n, b_{n-1}) \right] \pi_n(a_n|b_{n-1}) \right\} \quad (\text{III.91})$$

$$V_t^s(b_{t-1}) = \sup_{\pi_t(a_t|b_{t-1})} \left\{ \sum_{a_t \in \mathbb{A}_t} \left[\sum_{b_t \in \mathbb{B}_t} \left(\log \left(\frac{\mathbf{P}_t(b_t|b_{t-1}, a_t)}{\mathbf{P}_t^\pi(b_t|b_{t-1})} \right) \mathbf{P}_t(b_t|b_{t-1}, a_t) + V_{t+1}^s(b_t) \right) \mathbf{P}_t(b_t|b_{t-1}, a_t) - s\gamma_t^{UM}(a_t, b_{t-1}) \right] \pi_t(a_t|b_{t-1}) \right\}. \quad (\text{III.92})$$

Moreover, for a fixed initial distribution $\mathbf{P}_{B_{-1}}(b_{-1}) = \mu(b_{-1})$, then

$$C_{A^n \rightarrow B^n}^{FB,UMCO}(\kappa) = \inf_{s \geq 0} \left\{ \sum_{b_{-1}} V_0^s(b_{-1}) \mu(b_{-1}) + s(n+1)\kappa \right\}. \quad (\text{III.93})$$

The analogues of Theorem III.1 and Theorem III.2 are stated as a corollary.

Corollary III.1. (Necessary and sufficient conditions)

(a) The necessary and sufficient conditions for any input distribution $\{\pi_t(a_t|b_{t-1}) : t = 0, 1, \dots, n\}$ to achieve the supremum of the dynamic programming recursions (III.91) and (III.92) are the following.

For each $b_{n-1} \in \mathbb{B}_{n-1}$, there exist $V_n^s(b_{n-1})$ such that

$$V_n^s(b_{n-1}) = \sum_{b_n \in \mathbb{B}_n} \log \left(\frac{\mathbf{P}_n(b_n|a_n, b_{n-1})}{\mathbf{P}_n^\pi(b_n|b_{n-1})} \right) \mathbf{P}_n(b_n|a_n, b_{n-1}) - s\gamma_n^{UM}(a_n, b_{n-1}), \quad \forall a_n \in \mathbb{A}_n \text{ if } \pi_n(a_n|b_{n-1}) \neq 0, \quad (\text{III.94})$$

$$V_n^s(b_{n-1}) \leq \sum_{b_n \in \mathbb{B}_n} \log \left(\frac{\mathbf{P}_n(b_n|a_n, b_{n-1})}{\mathbf{P}_n^\pi(b_n|b_{n-1})} \right) \mathbf{P}_n(b_n|a_n, b_{n-1}) - s\gamma_n^{UM}(a_n, b_{n-1}), \quad \forall a_n \in \mathbb{A}_n \text{ if } \pi_n(a_n|b_{n-1}) = 0 \quad (\text{III.95})$$

and for each $t = 0, 1, \dots, n-1$ there exist $V_t^s(b_{t-1})$ such that

$$V_t^s(b_{t-1}) = \sum_{b_t \in \mathbb{B}_t} \left\{ \log \left(\frac{\mathbf{P}_t(b_t|a_t, b_{t-1})}{\mathbf{P}_t^\pi(b_t|b_{t-1})} \right) + V_{t+1}^s(b_t) \right\} \mathbf{P}_t(b_t|a_t, b_{t-1}) - s\gamma_t^{UM}(a_t, b_{t-1}), \quad \forall a_t \in \mathbb{A}_t, \text{ if } \pi_t(a_t|b_{t-1}) \neq 0, \quad (\text{III.96})$$

$$V_t^s(b_{t-1}) \leq \sum_{b_t \in \mathbb{B}_t} \left\{ \log \left(\frac{\mathbf{P}_t(b_t|a_t, b_{t-1})}{\mathbf{P}_t^\pi(b_t|b_{t-1})} \right) + V_{t+1}^s(b_t) \right\} \mathbf{P}_t(b_t|a_t, b_{t-1}) - s\gamma_t^{UM}(a_t, b_{t-1}), \quad \forall a_t \in \mathbb{A}_t, \text{ if } \pi_t(a_t|b_{t-1}) = 0. \quad (\text{III.97})$$

Moreover, $\{V_t^s(b_{t-1}) : (t, b_{t-1}) \in \{0, \dots, n\} \times \mathbb{B}_{t-1}\}$ is the value function defined by (III.90).

(b) The optimization problem $C_{A^n \rightarrow B^n}^{FB,UMCO}(\kappa)$ is non-nested and the value function is characterized by

$$V_t^s(b_{t-1}) = \sum_{i=t}^n \sup_{\pi_i(a_i|b_{i-1})} \mathbf{E}^\pi \left\{ \log \left(\frac{\mathbf{P}_i(B_i|B_{i-1}, A_i)}{\mathbf{P}_i^\pi(B_i|B_{i-1})} \right) - s\gamma^{UM}(A_i, B_{i-1}) \middle| B_{i-1} = b_{i-1} \right\} \quad (\text{III.98})$$

for all $(t, b_{t-1}) \in \{0, 1, \dots, n\} \times \mathbb{B}_{t-1}$ if and only if

$$\text{there exists constants } \{\bar{V}_t^s : t = 0, \dots, n\} \text{ such that } V_t^s(b_{t-1}) = \bar{V}_t^s, \quad \forall (t, b_{t-1}) \in \{0, 1, \dots, n\} \times \mathbb{B}_{t-1} \text{ which satisfy (III.94)-(III.97).} \quad (\text{III.99})$$

(b) If the channel distribution is time-invariant $\{\mathbf{P}(b_i|b_{i-1}, a_i) : i = 0, \dots, n\}$ and $\gamma_i^{UM}(\cdot, \cdot) = \gamma^{UM}(\cdot, \cdot) : i = 0, 1, \dots, n$, then the optimization problem $C_{A^n \rightarrow B^n}^{FB,UMCO}$ is non-nested and time-invariant, and the value function is characterized by

$$V_t^s(b_{t-1}) \equiv \bar{V}_t^s = (n-t+1) \sup_{\pi^{TI}(a_i|b_{i-1})} \mathbf{E}^{\pi^{TI}} \left\{ \log \left(\frac{\mathbf{P}(B_i|B_{i-1}, A_i)}{\mathbf{P}^\pi(B_i|B_{i-1})} \right) - s\gamma^{UM}(A_i, B_{i-1}) \middle| B_{i-1} = b_{i-1} \right\} \quad (\text{III.100})$$

where $\{\pi_i(a_i|b_{i-1}) = \pi^{TI}(a_i|b_{i-1}) : i = 0, \dots, n\}$ and $\{\mathbf{P}_i^\pi(b_i|b_{i-1}) = \mathbf{P}^{\pi^{TI}}(b_i|b_{i-1}) : i = 0, \dots, n\}$ are time-invariant, if and only if

there exists a constant \bar{V}_n^S such that $V_n(b_{n-1}) = \bar{V}_n$, $\forall b_{n-1} \in \mathbb{B}_{n-1}$ which satisfies (III.94), (III.95). (III.101)

Proof: The derivation is precisely as in Theorem III.1. ■

B. Necessary and Sufficient Conditions via Dynamic Programming: The Infinite Horizon case

In this section, we first identify sufficient conditions the convergence of the per unit time limit of the characterization of FTFI capacity, using the ergodic theory of Markov decision with randomized strategies, and infinite horizon dynamic programming. Then, we apply these to derive necessary and sufficient conditions for any channel input distribution to maximize the infinite horizon extremum problems $C_{A^\infty \rightarrow B^\infty}^{FB,UMCO}$ and $C_{A^\infty \rightarrow B^\infty}^{FB,UMCO}(\kappa)$.

For the material of this section we make the following assumption.

Assumptions III.1. (Time-Invariant or homogeneous)

The channel distribution and transmission cost function are time-invariant, and the optimal strategies are restricted to time-invariant strategies, i.e.,

$$\mathbf{P}_i(b_i|b_{i-1}, a_i) = \mathbf{P}(b_i|b_{i-1}, a_i), \quad \gamma_i^{UM}(a_i, b_{i-1}) \equiv \gamma^{UM}(a_i, b_{i-1}), \quad i = 0, \dots, n, \quad (\text{III.102})$$

$$\pi_i(a_i|b_{i-1}) = \pi^\infty(a_i|b_{i-1}), \quad i = 0, \dots, n \quad (\text{III.103})$$

and $\mathbb{A}_i = \mathbb{A}, \mathbb{B}_i = \mathbb{B}, i = 0, \dots, n$. Moreover, the initial distribution $\mathbf{P}_{B_{-1}} = \mu(b_{-1})$ is assumed fixed.

By invoking Assumptions III.1, we can introduce the corresponding extremum problem as follows. For fixed initial distribution $\mu(db_{-1}) \in \mathcal{M}(\mathbb{B})$, we define

$$J(\pi^\infty, \mu) \triangleq \liminf_{n \rightarrow \infty} \frac{1}{n} \mathbf{E}_\mu^{\pi^\infty} \left\{ \sum_{i=0}^{n-1} \log \left(\frac{\mathbf{P}(B_i|B_{i-1}, A_i)}{\mathbf{P}^{\pi^\infty}(B_i|B_{i-1})} \right) \right\} \equiv \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} I(A_i; B_i|B_{i-1}). \quad (\text{III.104})$$

By taking the supremum over all channel input distributions [27] and by using the fact that the alphabet spaces are of finite cardinality, we have the following identity.

$$J(\pi^{\infty,*}, \mu) \triangleq \sup_{\pi^\infty(a_i|b_{i-1}): i=0,1,\dots} J(\pi^\infty, \mu) \quad (\text{III.105})$$

$$= \liminf_{n \rightarrow \infty} \sup_{\pi^\infty(a_i|b_{i-1}): i=0,\dots,n} \frac{1}{n} \mathbf{E}_\mu^{\pi^\infty} \left\{ \sum_{i=0}^{n-1} \log \left(\frac{\mathbf{P}(B_i|B_{i-1}, A_i)}{\mathbf{P}^{\pi^\infty}(B_i|B_{i-1})} \right) \right\} \equiv C_{A^\infty \rightarrow B^\infty}^{FB,UMCO}. \quad (\text{III.106})$$

For abstract alphabet spaces the exchange of liminf and sup requires strong conditions [27]. Clearly, the above quantity $J(\pi^{\infty,*}, \mu)$ depends on the initial distribution $\mu(db_{-1})$.

Similarly, for a fixed initial state $B_{-1} = b_{-1}$ we also have the identity

$$\bar{J}(\pi^{\infty,*}, b_{-1}) \triangleq \sup_{\pi^\infty(a_i|b_{i-1}): i=0,1,\dots} \liminf_{n \rightarrow \infty} \frac{1}{n} \mathbf{E}_{b_{-1}}^{\pi^\infty} \left\{ \sum_{i=0}^{n-1} \log \left(\frac{\mathbf{P}(B_i|B_{i-1}, A_i)}{\mathbf{P}^{\pi^\infty}(B_i|B_{i-1})} \right) \right\} \quad (\text{III.107})$$

$$= \liminf_{n \rightarrow \infty} \sup_{\pi^\infty(a_i|b_{i-1}): i=0,\dots,n} \frac{1}{n} \mathbf{E}_{b_{-1}}^{\pi^\infty} \left\{ \sum_{i=0}^{n-1} \log \left(\frac{\mathbf{P}(B_i|B_{i-1}, A_i)}{\mathbf{P}^{\pi^\infty}(B_i|B_{i-1})} \right) \right\} \quad (\text{III.108})$$

which depends on the initial state b_{-1} . Note that Assumptions III.1 do not imply that the joint distribution of the process $\{A_0, B_0, A_1, B_1, \dots, A_n, B_n\}$ is stationary or that the marginal distribution of the output process $\{B_i : i = 0, \dots, n\}$ is stationary, because stationarity depends on the distribution of the initial state B_{-1} . However, it implies that the transition probabilities are time-invariant (i.e., homogeneous), hence, $\mathbf{P}_i^\pi(a_i, b_i|a_{i-1}, b_{i-1}) \equiv \mathbf{P}^{\pi^\infty}(a_i, b_i|a_{i-1}, b_{i-1}), \mathbf{P}_i^\pi(b_i|b_{i-1}) \equiv \mathbf{P}^{\pi^\infty}(b_i|b_{i-1}), i = 0, \dots, n$.

Next, we develop the material without imposing transmission cost constraints, because extensions to problems with transmission cost are easily obtained by using the material of the previous section.

1) *Sufficient Condition for Asymptotic Stationarity and Ergodicity from Finite-Time Dynamic Programming Recursions:* Consider the problem of maximizing the per unit time limiting version of $C_{A^n \rightarrow B^n}^{FB,UMCO}$, when the strategies are restricted to $\{\pi^\infty(a_i|b_{i-1}) : i = 0, \dots, n\}$. From the previous section, the finite horizon value function satisfies the dynamic programming equation

$$V_t(b_{t-1}) = \sup_{\pi^\infty(\cdot|b_{t-1})} \left\{ \sum_{a_t} \left\{ \sum_{b_t} \log \left(\frac{\mathbf{P}(b_t|b_{t-1}, a_t)}{\mathbf{P}\pi^\infty(b_t|b_{t-1})} \right) \mathbf{P}(b_t|b_{t-1}, a_t) + \sum_{b_t} V_{t+1}(b_t) \mathbf{P}(b_t|b_{t-1}, a_t) \right\} \pi^\infty(a_t|b_{t-1}) \right\}. \quad (\text{III.109})$$

Since b_{t-1} is always fixed, we let $V_t(b_{t-1}) = V_t(b_{-1}), t = 0, \dots, n-1$. Since the transition probabilities are time-invariant, we can define, for simplicity, the variables $\tilde{V}_t(b_{-1}) = V_{n-t}(b_{-1}), t = 0, \dots, n-1$. Then $\{\tilde{V}_t(\cdot) : t = 1, \dots, n\}$ satisfy the following equation.

$$\begin{aligned} \tilde{V}_t(b_{-1}) = \sup_{\pi^\infty(\cdot|b_{-1})} & \left\{ \sum_{a_0 \in \mathbb{A}} \left\{ \sum_{b_0 \in \mathbb{B}} \log \left(\frac{\mathbf{P}(b_0|b_{-1}, a_0)}{\mathbf{P}\pi^\infty(b_0|b_{-1})} \right) \mathbf{P}(b_0|b_{-1}, a_0) \right. \right. \\ & \left. \left. + \sum_{b_0 \in \mathbb{B}} \tilde{V}_{t-1}(b_0) \mathbf{P}(b_0|b_{-1}, a_0) \right\} \pi^\infty(a_0|b_{-1}) \right\}, \quad t \in \{1, \dots, n\}. \end{aligned} \quad (\text{III.110})$$

Next, we introduce a sufficient condition to test whether the per unit time limit of the solution to the dynamic programming recursions exists and it is independent of the initial state $B_{-1} = b_{-1} \in \mathbb{B}$.

Assumptions III.2. (Sufficient condition for convergence of dynamic programming recursions)

Assume that there exists a $V : \mathbb{B} \mapsto \mathbb{R}$, and a $J^* \in \mathbb{R}$ such that for all $b_{-1} \in \mathbb{B}$

$$\lim_{t \rightarrow \infty} (\tilde{V}_t(b_{-1}) - tJ^*) = V(b_{-1}). \quad (\text{III.111})$$

Clearly, if Assumptions III.2 hold, then $\lim_{t \rightarrow \infty} \frac{1}{t} \tilde{V}_t(b_{-1}) = J^*, \forall b_{-1} \in \mathbb{B}$ (because for finite alphabet spaces, the dynamic programming operator maps bounded continuous functions to bounded continuous functions) [28]. This means the per unit time limit of the dynamic programming recursion is independent of the initial state b_{-1} , which then implies $C_{A^n \rightarrow B^n}^{FB,UMCO}$ is independent of the choice of the initial distribution $\mu(b_{-1})$.

Remark III.2. (Test of asymptotic stationarity and ergodicity)

Given any channel we can verify that the optimal channel input distribution induces asymptotic stationarity and ergodicity of the corresponding joint process $\{(A_i, B_i) : i = 0, 1, \dots\}$ by solving the dynamic programming recursions analytically for finite “ n ” via (III.109), and then identifying conditions on the channel parameters so that Assumptions III.2 hold.

In view of Assumptions III.2, we have the following lemma.

Lemma III.1. If Assumptions III.2 hold and there exists a $\{\pi^{\infty,*}(a_0|b_{-1}) \in \mathcal{M}(\mathbb{A}) : b_{-1} \in \mathbb{B}\}$ and a corresponding pair $\left\{ (V(b_{-1}), J^*) : b_{-1} \in \mathbb{B}, J^* \in \mathbb{R} \right\}$, which solves

$$J^* + V(b_{-1}) = \sup_{\pi^\infty(a_0|b_{-1})} \left\{ \sum_{a_0} \left\{ \sum_{b_0} \log \left(\frac{\mathbf{P}(b_0|b_{-1}, a_0)}{\mathbf{P}\pi^\infty(b_0|b_{-1})} \right) \mathbf{P}(b_0|b_{-1}, a_0) + \sum_{b_0} V(b_0) \mathbf{P}(b_0|b_{-1}, a_0) \right\} \pi^\infty(a_0|b_{-1}) \right\}. \quad (\text{III.112})$$

then feedback capacity is given by

$$J^* = C_{A^\infty \rightarrow B^\infty}^{FB,UMCO} \triangleq \liminf_{n \rightarrow \infty} \frac{1}{n} \sup_{\pi^\infty(\cdot|b_{i-1})} \mathbf{E}^{\pi^\infty} \left\{ \sum_{i=0}^{n-1} \log \left(\frac{\mathbf{P}(B_i|B_{i-1}, A_i)}{\mathbf{P}\pi^\infty(B_i|B_{i-1})} \right) \right\}, \quad \forall \mu(db_{-1}) \in \mathcal{M}(\mathbb{B}) \quad (\text{III.113})$$

and moreover the value $C_{A^\infty \rightarrow B^\infty}^{FB,UMCO}$ does not depend on the choice of the initial distribution $\mu(db_{-1}) \in \mathcal{M}(\mathbb{B})$.

Proof: See Appendix A. ■

Thus, we have two different ways to determine sufficient conditions for J^* to correspond to feedback capacity; one based on Remark III.2, and one based on Lemma III.1, i.e., by solving the infinite horizon dynamic programming equation (III.112).

Next, we state the necessary and sufficient conditions for any $\{\pi^\infty(a_0|b_{-1}) \in \mathcal{M}(\mathbb{A}) : b_{-1} \in \mathbb{B}\}$ to be a solution of the dynamic programming equation (III.112).

Theorem III.3. (*Infinite horizon Necessary and Sufficient conditions*)

Suppose Assumptions III.2 hold and there exists a $\{\pi^{\infty,*}(a_0|b_{-1}) \in \mathcal{M}(\mathbb{A}) : b_{-1} \in \mathbb{B}, J^* \in \mathbb{R}\}$ and a corresponding pair $\left\{ \left(V(b_{-1}), J^* \right) : b_{-1} \in \mathbb{B} \right\}$, which solves (III.112).

The necessary and sufficient conditions for any input distribution $\{\pi^\infty(a_0|b_{-1}) \in \mathcal{M}(\mathbb{A}) : b_{-1} \in \mathbb{B}\}$ to achieve the supremum of the dynamic programming equation (III.111) are the following.

There exist $\{V(b_{-1}) : b_{-1} \in \mathbb{B}\}$ such that

$$J^* + V(b_{-1}) = \sum_{b_0} \left(\log \left(\frac{\mathbf{P}(b_0|a_0, b_{-1})}{\mathbf{P}^{\pi^\infty}(b_0|b_{-1})} \right) + V(b_0) \right) \mathbf{P}(b_0|a_0, b_{-1}), \quad \forall a_0 \in \mathbb{A} \text{ if } \pi^\infty(a_0|b_{-1}) \neq 0, \quad (\text{III.114})$$

$$J^* + V(b_{-1}) \leq \sum_{b_0} \left(\log \left(\frac{\mathbf{P}(b_0|a_0, b_{-1})}{\mathbf{P}^{\pi^\infty}(b_0|b_{-1})} \right) + V(b_0) \right) \mathbf{P}(b_0|a_0, b_{-1}), \quad \forall a_0 \in \mathbb{A} \text{ if } \pi^\infty(a_0|b_{-1}) = 0. \quad (\text{III.115})$$

Moreover, $\{V(b_{-1}) : b_{-1} \in \mathbb{B}\}$ is the value function defined by (III.112).

Proof: Consider the dynamic programming equation (III.112) and repeat the necessary steps of the derivation of Theorem III.1. A more direct approach is to use the necessary and sufficient conditions of Theorem III.1, as follows. Re-writing the necessary and sufficient conditions (III.72), (III.73) as done in (A.186), using Assumptions III.2, to verify that (III.114), (III.115) are the resulting equations. ■

C. Sufficient Conditions for Asymptotic Stationarity and Ergodicity based on Irreducibility

In this section we give another set of assumptions based on irreducibility of the channel output transition probability for each channel input conditional distribution.

Define

$$\bar{\ell}(b_{i-1}, a_i) \triangleq \mathbf{E}^{\pi^\infty} \left\{ \log \left(\frac{\mathbf{P}(B_i|B_{i-1}, A_i)}{\mathbf{P}^{\pi^\infty}(B_i|B_{i-1})} \right) \middle| B_{i-1} = b_{i-1}, A_i = a_i \right\} \quad (\text{III.116})$$

$$\equiv \sum_{b_i \in \mathbb{B}} \log \left(\frac{\mathbf{P}(b_i|b_{i-1}, a_i)}{\mathbf{P}^{\pi^\infty}(b_i|b_{i-1})} \right) \mathbf{P}(b_i|b_{i-1}, a_i), \quad (\text{III.117})$$

$$\ell(b_{i-1}, \pi^\infty(b_{i-1})) \triangleq \mathbf{E}^{\pi^\infty} \left\{ \log \left(\frac{\mathbf{P}(B_i|B_{i-1}, A_i)}{\mathbf{P}^{\pi^\infty}(B_i|B_{i-1})} \right) \middle| B_{i-1} = b_{i-1} \right\} \equiv \sum_{a_i \in \mathbb{A}} \bar{\ell}(b_{i-1}, a_i) \pi^\infty(a_i|b_{i-1}). \quad (\text{III.118})$$

To apply standard results of the Markov Decision (MD) theory from [27], [28], we introduce the following notation. Each element of the alphabet space \mathbb{B} is identified by the vector $\mathbb{B} = \{b(1), \dots, b(|\mathbb{B}|)\}$, where $|\mathbb{B}|$ is the cardinality of the set \mathbb{B} . Then we can identify any $V : \mathbb{B} \mapsto \mathbb{R}$ with a vector in $\mathbb{R}^{|\mathbb{B}|}$. Similarly, any channel input distribution is identified with

$$\pi^\infty \triangleq \left\{ \pi^\infty(b(1)), \pi^\infty(b(2)), \dots, \pi^\infty(b(|\mathbb{B}|)) \right\} \triangleq \left\{ \pi^\infty(\cdot|b(i)) \in \mathcal{M}(\mathbb{A}) : b(i) \in \mathbb{B} \right\}. \quad (\text{III.119})$$

Next, we define the vector pay-off and channel output transition probability matrix as follows.

$$\ell(\pi^\infty) \triangleq \left(\ell(b(1), \pi^\infty(b(1))) \quad \dots \quad \ell(b(|\mathbb{B}|), \pi^\infty(b(|\mathbb{B}|))) \right)^T \in \mathbb{R}^{|\mathbb{B}|}, \quad (\text{III.120})$$

$$\mathbf{P}(\pi^\infty) = \left\{ \mathbf{P}^{\pi^\infty}(b_i | b_{i-1}) : (b_i, b_{i-1}) \in \mathbb{B} \times \mathbb{B} \right\} \in \mathbb{R}^{|\mathbb{B}| \times |\mathbb{B}|}. \quad (\text{III.121})$$

Let $\{\mu(b(i)) : i = 1, 2, \dots, |\mathbb{B}|\} \in \mathbb{R}^{|\mathbb{B}|}$ be defined by $\mu(b(i)) = \mathbf{P}(B_{-1} = b(i)), i = 1, \dots, |\mathbb{B}|$.

Using the above notation we have the following main theorem.

Theorem III.4. (Dynamic programming equation under irreducibility)

Suppose Assumptions III.1 holds and for each channel input distribution π^∞ , the transition probability matrix of the output process $\mathbf{P}(\pi^\infty) \equiv \{\mathbf{P}^{\pi^\infty}(b_0 | b_{-1}) : (b_0, b_{-1}) \in \mathbb{B} \times \mathbb{B}\}$ is irreducible.

Then for any channel input distribution π^∞ the expression (III.104) is given by

$$J(\pi^\infty, \mu) = \mathbf{v}(\pi^\infty)^T \ell(\pi^\infty) \equiv J(\pi^\infty) \quad (\text{III.122})$$

i.e., it is independent of $\mu(\cdot)$, where $\mathbf{v}(\pi^\infty)$ is the unique invariant probability distribution of the channel output process $\{B_0, B_1, \dots\}$, which satisfies

$$\mathbf{P}(\pi^\infty) \mathbf{v}(\pi^\infty) = \mathbf{v}(\pi^\infty). \quad (\text{III.123})$$

If there exists a time-invariant Markov channel distribution $\pi^\infty(\cdot | \cdot)$ such that

$$J(\pi^{\infty,*}) = \max_{\pi^\infty} J(\pi^\infty)$$

then there exists a pair $(V(\pi^{\infty,*}, \cdot), J(\pi^{\infty,*}))$, $V(\pi^{\infty,*}, \cdot) : \mathbb{B} \mapsto \mathbb{R}^{|\mathbb{B}|}$ and $J(\pi^\infty) \in \mathbb{R}$ that is a solution of the dynamic programming equation

$$J(\pi^{\infty,*}) + V(\pi^{\infty,*}, b_{-1}) = \sup_{\pi^\infty(\cdot | b_{-1})} \left\{ \ell(b_{-1}, \pi^\infty(b_{-1})) + \sum_{z \in \mathbb{B}} V(\pi^{\infty,*}, z) \mathbf{P}^{\pi^\infty}(z | b_{-1}) \right\}. \quad (\text{III.124})$$

Moreover, $J^* \equiv J(\pi^{\infty,*}) = C_{A^\infty \rightarrow B^\infty}^{FB, UMC}$ satisfies (III.113) and corresponds to feedback capacity.

Proof: This is shown in Appendix B. ■

We make the following comments.

Remark III.3. (Comments on Theorem III.4)

(a) Theorem III.4 gives sufficient conditions in terms of irreducibility of channel output transition probability matrix $\mathbf{P}(\pi^\infty)$ to test whether the per unit time limit of the FTFI capacity corresponds to feedback capacity. Unfortunately, it is not possible to know prior to solving the dynamic programming equation (III.124) whether the irreducibility condition holds, because the transition probability $\mathbf{P}(\pi^\infty)$ is a functional of the optimal channel input distribution. A similar issue occurs in the analysis provided by Chen and Berger [15, Lemma 2, Theorem 3]. In view of this technicality it is more appropriate to apply the necessary and sufficient conditions of Theorem III.1 to determine the optimal channel input distribution and corresponding characterization of FTFI capacity, and then follow the suggestion given under Remark III.2.

(b) The solution of V obtained from (III.124) is unique up to an additive constant, and if $\pi^*(\cdot | b_{-1})$ attains the maximum in (III.124) for every b_{-1} , then $\pi^*(\cdot | \cdot)$ is an optimal channel input distribution, and the maximum cost is J^* .

(c) In specific application examples it may happen that the optimal channel input probability distribution $\pi^{\infty,*}(\cdot | \cdot)$ induces a transition probability matrix $\mathbf{P}(\pi^{\infty,*})$ which is reducible, i.e., not irreducible. For completeness, this specific case is addressed in Remark III.4.

Next, we provide an iterative algorithm to compute the optimal channel input distribution and the feedback

capacity. In Section IV-D1, we illustrate how Algorithm 1 is implemented through an example.

Algorithm 1

1) Let $m = 0$ and select an arbitrary stationary Markov channel input symbol distribution π_0 .

2) Solve the equation

$$J(\pi_m)e + V(\pi_m) = \ell(\pi_m) + V^T(\pi_m)\mathbf{P}(\pi_m), \quad e \triangleq (1, \dots, 1) \in \mathbb{R}^{|\mathbb{B}|} \quad (\text{III.125})$$

for $J(\pi_m) \in \mathbb{R}$ and $V(\pi_m) \in \mathbb{R}^{|\mathbb{B}|}$.

3) Let

$$\pi_{m+1} = \operatorname{argmax}_{\pi} \left\{ \ell(\pi) + V^T(\pi)\mathbf{P}(\pi) \right\}. \quad (\text{III.126})$$

4) If $\pi_{m+1} = \pi_m$, let $\pi^* = \pi_m$; else let $m = m + 1$ and return to step 2.

Remark III.4. Theorem III.4 and Algorithm 1 pre-suppose that we know in advance that the transition probability matrix $\mathbf{P}(\pi^\infty)$ of the channel output process, when evaluated at the optimal strategy $\pi^{\infty,*}(\cdot|\cdot)$ is irreducible. If irreducibility does not hold, then the dynamic programming equation (III.124) may not be sufficient to give the optimal channel input distribution and the feedback capacity. In particular, if $\mathbf{P}(\pi^\infty)$ is reducible then (III.124) need not have a solution. To overcome this limitation an additional equation is added to (III.124) giving the following pair of equations.

$$J^*(b_{-1}) = \sup_{\pi^\infty(\cdot|b_{-1})} \left\{ \int_{\mathbb{A} \times \mathbb{B}} J^*(b_0) \mathbf{P}^{\pi^\infty}(b_0|b_{-1}) \right\} \quad (\text{III.127})$$

$$J^*(b_{-1}) + V(b_{-1}) = \sup_{\pi^\infty(\cdot|b_{-1})} \left\{ \int_{\mathbb{A} \times \mathbb{B}} \left\{ \log \left(\frac{\mathbf{P}(b_0|a_0, b_{-1})}{\mathbf{P}^{\pi^\infty}(b_0|b_{-1})} \right) + V(b_0) \right\} \mathbf{P}^{\pi^\infty}(b_0|b_{-1}) \right\}. \quad (\text{III.128})$$

We refer to the pair (III.127) and (III.128) as the generalized dynamic programming equations. The proposed pair of dynamic programming equations completely characterize feedback capacity.

D. Error exponents for the UMC Channel with feedback

In this section, we provide bounds on the probability of error of maximum likelihood decoding, by utilizing the results in [25] and [29]. However, we go one step further and show how to compute this bound, taking advantage of the structure of the capacity achieving distribution.

Consider the channel $\{\mathbf{P}_i(b_i|b_{i-1}, a_i) : i = 0, \dots, n\}$, where $\mathbb{B}_i = \mathbb{B}, \mathbb{A}_i = \mathbb{A}, i = 0, \dots, n$. Let $\mathbf{P}_{e,m}^{(n)}(b_{-1})$ denote the probability of error for an arbitrary message $m \in \mathcal{M}_n \triangleq \{1, 2, \dots, M_n = \lfloor 2^{nR} \rfloor\}$, given the initial state $b_{-1} \in \mathbb{B}$. From [29] there exists a feedback code for which the probability of error is bounded above as follows.⁴

$$\mathbf{P}_{e,m}^{(n)}(b_{-1}) \leq 4|\mathbb{B}|2^{\{-n[-\rho R + \bar{F}_n(\rho)]\}}, \quad \forall m \in \mathcal{M}_n, \quad b_{-1} \in \mathbb{B}_{-1}, \quad 0 \leq \rho \leq 1, \quad (\text{III.129})$$

$$\bar{F}_n(\rho) \triangleq \frac{-\rho \log |\mathbb{B}|}{n} + \max_{\mathbf{P}_i(a_i|a^{i-1}, b^{i-1}) : i=0,1,\dots,n} \left[\min_{b_{-1} \in \mathbb{B}} E_{0,n}^{\mathbf{P}}(\rho, b_{-1}) \right] \quad (\text{III.130})$$

$$E_{0,n}^{\mathbf{P}}(\rho, b_{-1}) = -\frac{1}{n} \log \sum_{(b_0, \dots, b_{n-1})} \left[\sum_{(a_0, \dots, a_{n-1})} \prod_{i=0}^{n-1} \mathbf{P}_i(a_i|a^{i-1}, b^{i-1}) \mathbf{P}_i(b_i|a_i, b_{i-1})^{\frac{1}{1+\rho}} \right]^{1+\rho}. \quad (\text{III.131})$$

However, by restricting the channel input distribution in (III.130), (III.131), to the set $\{\pi_i(da_i|b_{i-1}) : i =$

⁴If the initial state is known both to the encoder and the decoder then the cardinality of the state alphabet, $|\mathbb{B}|$, in (III.129) and (III.130) are removed [Problem 5.37, [25]].

$0, \dots, n\} \in \overset{\circ}{\mathcal{P}}_{[0,n]}^{FB}$, the following upper bound is obtained.

$$\mathbf{P}_{e,m}^{(n)}(b_{-1}) \leq 4|\mathbb{B}|2^{\{-n[-\rho R + F_n(\rho)]\}}, \quad \forall m \in \mathcal{M}_n, \quad b_{-1} \in \mathbb{B}_{-1}, \quad 0 \leq \rho \leq 1, \quad (\text{III.132})$$

$$F_n(\rho) \triangleq \frac{-\rho \log |\mathbb{B}|}{n} + \min_{b_{-1} \in \mathbb{B}} E_{0,n}^\pi(\rho, b_{-1}), \quad \forall \{\pi_i(da_i|b_{i-1}) : i = 0, \dots, n\} \in \overset{\circ}{\mathcal{P}}_{[0,n]}^{FB}, \quad (\text{III.133})$$

$$E_{0,n}^\pi(\rho, b_{-1}) = -\frac{1}{n} \log \sum_{(b_0, \dots, b_{n-1})} \left[\sum_{(a_0, \dots, a_{n-1})} \prod_{i=0}^{n-1} \pi_i(a_i|b_{i-1}) \mathbf{P}_i(b_i|a_i, b_{i-1})^{\frac{1}{1+\rho}} \right]^{1+\rho}. \quad (\text{III.134})$$

Next, we derive simplified equations for (III.132) -(III.134), in order to compute the bound on the probability of error. For the rest of the analysis we view the memory of the channel on the previous output symbol as the state of the channel, defined by $s_{i-1} \triangleq b_{i-1}, i = 0, 1, \dots, n-1$. Then we transform the channel to an equivalent channel of the form $\mathbf{P}_i(b_i|a_i, b_{i-1}) = \mathbf{P}_i(b_i|a_i, s_{i-1}), i = 0, \dots, n$. Since the state of the channel is known at the decoder, we apply the methodology used to derive Theorem 5.9.3, [25] and (III.134), to obtain an upper bound on the probability of error, which is computationally less intensive than (III.132), as follows. At each time i , the channel distribution is further transformed to

$$\mathbf{P}_i(b_i, s_i|a_i, s_{i-1}) = \begin{cases} \mathbf{P}_i(b_i|a_i, s_{i-1}) & \text{if } s_i = b_i \\ 0 & \text{otherwise.} \end{cases} \quad i = 0, \dots, n. \quad (\text{III.135})$$

Substituting (III.135) into (III.134) gives the following equivalent expression.

$$E_{0,n}^\pi(\rho, b_{-1}) = -\frac{1}{n} \log \sum_{(b_0, \dots, b_{n-1})} \left[\sum_{(a_0, \dots, a_{n-1})} \prod_{i=0}^{n-1} \pi_i(a_i|b_{i-1}) \mathbf{P}_i(b_i|a_i, s_{i-1})^{\frac{1}{1+\rho}} \right]^{1+\rho} \quad (\text{III.136})$$

$$= -\frac{1}{n} \log \sum_{(s_0, \dots, s_{n-1})} \sum_{(b_0, \dots, b_{n-1})} \left[\sum_{(a_0, \dots, a_{n-1})} \prod_{i=0}^{n-1} \pi_i(a_i|s_{i-1}) \mathbf{P}_i(b_i, s_i|a_i, s_{i-1})^{\frac{1}{1+\rho}} \right]^{1+\rho} \quad (\text{III.137})$$

$$= -\frac{1}{n} \log \sum_{(s_0, \dots, s_{n-1})} \prod_{i=0}^{n-1} \sum_{b_i} \left[\sum_{a_i} \pi_i(a_i|s_{i-1}) \mathbf{P}_i(b_i, s_i|a_i, s_{i-1})^{\frac{1}{1+\rho}} \right]^{1+\rho}. \quad (\text{III.138})$$

Define the inner summations in (III.138) by

$$\Lambda_i^\pi(s_i, s_{i-1}) \triangleq \sum_{b_i} \left[\sum_{a_i} \pi_i(a_i|s_{i-1}) \mathbf{P}_i(b_i, s_i|a_i, s_{i-1})^{\frac{1}{1+\rho}} \right]^{1+\rho}, \quad i = 0, \dots, n. \quad (\text{III.139})$$

Then, by substituting (III.139) in (III.138), we obtain

$$E_{0,n}^\pi(\rho, b_{-1}) = -\frac{1}{n} \log \sum_{(s_0, \dots, s_n)} \prod_{i=0}^{n-1} \Lambda_i^\pi(s_i, s_{i-1}). \quad (\text{III.140})$$

Let $\{\Lambda_i^\pi(s_i, s_{i-1}) : (s_i, s_{i-1}) \in \mathbb{B} \times \mathbb{B}\}$ denote the matrix with elements identified by $\Lambda_i^\pi(s_i, s_{i-1}), s_i = 1, \dots, |\mathbb{B}|, s_{i-1} = 1, \dots, |\mathbb{B}|$, that is, the matrix is denoted by

$$[\Lambda_i^\pi(s_i, s_{i-1})] \triangleq \begin{bmatrix} \Lambda_i^\pi(1, 1) & \dots & \Lambda_i^\pi(1, |\mathbb{B}|) \\ \vdots & \ddots & \vdots \\ \Lambda_i^\pi(|\mathbb{B}|, 1) & \dots & \Lambda_i^\pi(|\mathbb{B}|, |\mathbb{B}|) \end{bmatrix}, \quad i = 0, \dots, n. \quad (\text{III.141})$$

The computation of the error probability is difficult, in view of the time varying properties of the channel distribution and the channel input distribution, which implies the matrix $[\Lambda_i^\pi(s_i, s_{i-1})]$ is also time-varying. However, by following the derivation of equation (5.9.45) in [25], we derive the following bound on the probability of error for the UMCO channel with feedback.

Theorem III.5. (Error probability bound for maximum likelihood decoding)

Suppose the channel distribution is time-invariant given by $\{\mathbf{P}(b_i|b_{i-1}, a_i) : i = 0, \dots, n\}$ and the proba-

bility of error defined by (III.132)-(III.134) is evaluated at any time-invariant channel input distribution $\{\pi^{TI}(a_i|b_{i-1}) : i = 0, \dots, n\}$. Then

- (i) The matrix $[\Lambda_i^\pi(s_i, s_{i-1})] = [\Lambda^{\pi^{TI}}(s_i, s_{i-1})]$, $i = 0, \dots, n$ is time-invariant.
- (ii) If the time-invariant matrix $[\Lambda^{\pi^{TI}}(s_i, s_{i-1})]$ is irreducible, then there exists a feedback code for which the probability of error is bounded above as follows.

$$\mathbf{P}_{e,m}^{(n)} \leq 4|\mathbb{B}| \frac{v_{\max}}{v_{\min}} 2^{\left\{-n \left[-\rho R - \log \lambda_{\max}^{\pi^{TI}}(\rho)\right]\right\}}, \quad \forall m \in \mathcal{M}_n, \quad 0 \leq \rho \leq 1, \quad (\text{III.142})$$

where $\lambda_{\max}^{\pi^{TI}}(\rho)$ is the largest eigenvalue of the matrix $[\Lambda^{\pi^{TI}}(s_i, s_{i-1})]$, and v_{\max} and v_{\min} are the maximum and minimum components, respectively, of the positive eigenvector that corresponds to the largest eigenvalue.

Proof: (i) The first statement is due to the assumptions and follows directly from the fact that $E_{0,n}^\pi(\rho, b_{-1}) = E_{0,n}^{\pi^{TI}}(\rho, b_{-1}) \triangleq -\frac{1}{n} \log \sum_{(s_0, \dots, s_n)} \prod_{i=0}^{n-1} \Lambda^{\pi^{TI}}(s_i, s_{i-1})$.
(ii) For an irreducible matrix $[\Lambda(s_i, s_{i-1})]$ with non negative components we can apply the Frobenius theorem, to show that the following inequality holds [25].

$$\left| E_{0,n}^{\pi^{TI}}(\rho, b_{-1}) + \log \lambda_{\max}^{\pi^{TI}}(\rho) \right| \leq \frac{1}{n} \log \frac{v_{\max}}{v_{\min}}. \quad (\text{III.143})$$

The upper bound (III.142) follows from the last expression. ■

Note that the probability of error in Theorem III.5 is independent of the initial state $b_{-1} \in \mathbb{B}$. In Section IV-A3 we evaluate (III.142) of Theorem III.5 for a specific channel with memory.

IV. THE BSSC WITH & WITHOUT FEEDBACK AND WITH & WITHOUT TRANSMISSION COST

In this section, we apply the main results of the previous section to the unit memory channel Binary State Symmetric Channel (BSSC) defined by

$$\mathbf{P}(b_i|a_i, b_{i-1}) = \begin{matrix} & \begin{matrix} 0,0 & 0,1 & 1,0 & 1,1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{bmatrix} \alpha & \beta & 1-\beta & 1-\alpha \\ 1-\alpha & 1-\beta & \beta & \alpha \end{bmatrix} \end{matrix}, \quad i = 0, 1, 2, \dots, n, \quad (\alpha, \beta) \in [0, 1] \times [0, 1]. \quad (\text{IV.144})$$

We show using Theorem III.2, that the feedback capacity optimization problem is non-nested and the optimal channel input distribution is time invariant. Further, we derive explicit expressions for feedback capacity and capacity without feedback, and we show that the capacity achieving distribution and the corresponding transition probability of the channel output processes are characterized by doubly stochastic matrices. Moreover, we show that feedback does not increase capacity, and that capacity without feedback is achieved by a first order Markov channel input distribution, which is also doubly stochastic.

First we show that the BSSC, is equivalent to a channel with state information $s_i \triangleq a_i \oplus b_{i-1}$, $i = 0, 1, \dots, n$, where \oplus denotes the modulo2 addition, as depicted in Fig. IV.1. Clearly, this transformation is one to one and onto, i.e., for a fixed channel input symbol value a_i (respectively channel output symbol value b_{i-1}) then s_i is uniquely determined by the value of b_{i-1} (respectively a_i) and vice-versa. Hence, we

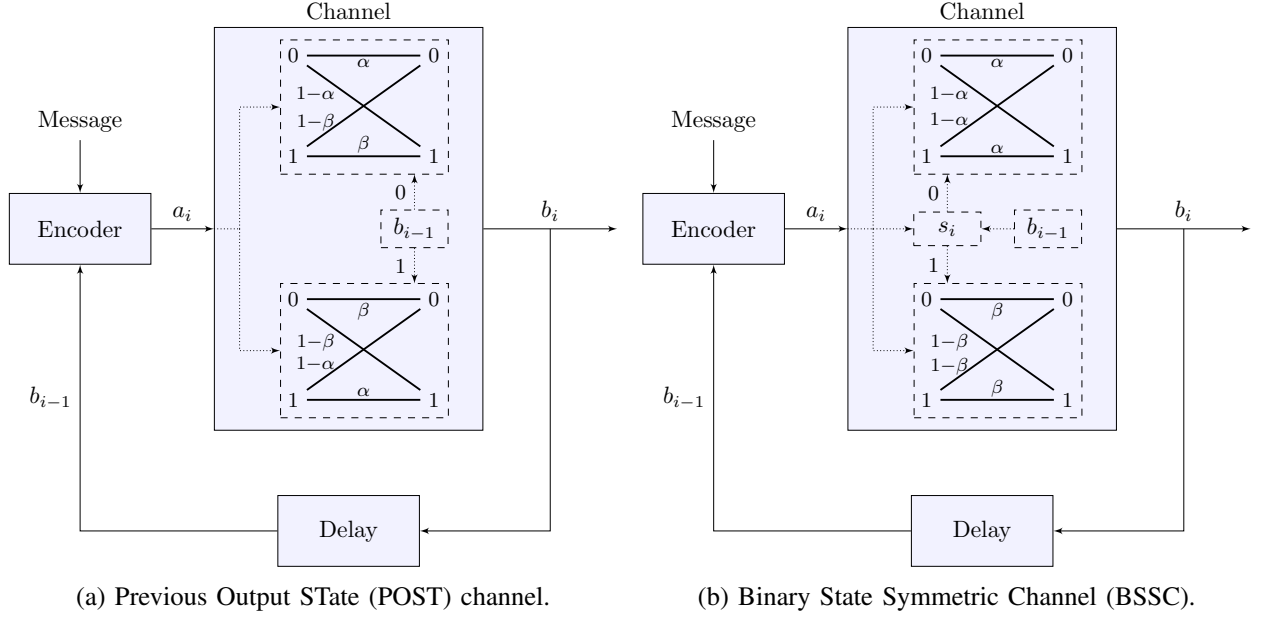


Fig. IV.1: An equivalent model.

obtain the following equivalent representation of the BSSC.

$$\mathbf{P}(b_i|a_i, s_i = 0) = \begin{bmatrix} \alpha & 1 - \alpha \\ 1 - \alpha & \alpha \end{bmatrix}, \quad i = 0, 1, \dots, n, \quad (\text{IV.145})$$

$$\mathbf{P}(b_i|a_i, s_i = 1) = \begin{bmatrix} \beta & 1 - \beta \\ 1 - \beta & \beta \end{bmatrix}, \quad i = 0, 1, \dots, n. \quad (\text{IV.146})$$

The above transformation highlights the symmetric form of the BSSC, since, for a fixed state $s_i \in \{0, 1\}$, the channel decomposes (IV.144) into two Binary Symmetric Channels (BSC), with transition probabilities given by (IV.145) and (IV.146), respectively. Therefore, for a fixed value of previous output symbol, b_{i-1} , the encoder by choosing the current input symbol, a_i , knows which of the two BSC's is applied at each transmission time. This decomposition motivates the name state-symmetric channel.

The following notation will be used in the rest of the paper.

- $\text{BSSC}(\alpha, \beta)$ denotes the BSSC with transition probabilities defined by (IV.144);
- $\text{BSC}(1 - \alpha)$ denotes the “state zero” channel defined by (IV.145);
- $\text{BSC}(1 - \beta)$ denotes the “state one” channel defined by (IV.146).

The necessity of imposing transmission cost constraint on the channel, is discussed by Shannon in [30, pp. 162–163] and it is encapsulated in the following statement. “There is a curious and provocative duality between the properties of a source with a distortion measure and those of a channel. This duality is enhanced if we consider channels in which there is a “cost” associated with the different input letters, and it is desired to find the capacity subject to the constraint that the expected cost not exceed a certain quantity...”. In [19], it is shown that the BSSC is in perfect duality with the Binary Symmetric Markov Source (BSMS) with respect to a transmission cost function for the channel and a fidelity constraint for the source. This is a generalization of JSCM of the discrete memoryless Bernoulli source with single letter Hamming distortion transmitted over a memoryless BSC.

Next, we illustrate that the cost constraint is natural when imposed on the BSSC. The memory on the previous output symbols and its decomposable nature, allow us to impose a cost function related to the

state of the channel.

The physical interpretation of the transmission cost is the following. The two states of the BSSC are

- $s_i = 0$ which is defined as the “state zero” channel and corresponds to a BSC with crossover probability $(1 - \alpha)$;
- $s_i = 1$ which is defined as the “state one” channel and corresponds to a BSC with crossover probability $(1 - \beta)$;

Suppose $\alpha > \beta \geq 0.5$. Then the capacity of the state zero channel is greater than the capacity of the state one channel. With “abuse” of terminology, the state zero channel is interpreted as the “good channel” and the state one channel is interpreted as the “bad channel”. With such interpretation it is reasonable to impose a higher cost, when employing the “good channel”, and a lower cost, when employing the “bad channel”. This policy is quantified by assigning a binary pay-off equal to “1”, that is, when the good channel is used, and a pay-off equal to “0”, that is, when the bad channel is used.

Definition IV.1. (Binary cost function for the BSCC)

The cost function of the BSSC satisfies

$$\gamma(a_i, b_{i-1}) = \overline{a_i \oplus b_{i-1}} \triangleq \begin{cases} 1 & \text{if } a_i = b_{i-1} \ (s_i = 0) \\ 0 & \text{if } a_i \neq b_{i-1} \ (s_i = 1) \end{cases} \quad i = 0, 1, \dots, n. \quad (\text{IV.147})$$

The average transmission cost constraint is defined by

$$\frac{1}{n+1} \mathbf{E} \left\{ \sum_{i=0}^n \gamma(A_i, B_{i-1}) \right\} \leq \kappa, \quad \kappa \in [0, \kappa_{\max}] \quad (\text{IV.148})$$

where the letter-by-letter average transmission cost is given by

$$\mathbf{E} \{ \gamma(A_i, B_{i-1}) \} = \mathbf{P}_i(a_i = 0, b_{i-1} = 0) + \mathbf{P}_i(a_i = 1, b_{i-1} = 1) = \mathbf{P}_i(s_i = 0). \quad (\text{IV.149})$$

This cost function may differ, according to someone’s preferences. For example, if we want to penalize the use of the “bad” channel, we may employ the complement of the cost function (IV.147). A more general cost function is

$$\gamma(a_i, b_{i-1}) \triangleq \begin{cases} \bar{\gamma} & \text{if } a_i = b_{i-1} \ (s_i = 0) \\ 1 - \bar{\gamma} & \text{if } a_i \neq b_{i-1} \ (s_i = 1) \end{cases} \quad i = 0, 1, \dots, n \quad (\text{IV.150})$$

where $\bar{\gamma} \in [0, 1]$. However, the binary form of the transmission cost does not downgrade the problem, since, the average cost is a linear functional, and it can be easily upgraded to more complex forms, without affecting the proposed methodology.

Additional observations regarding the above formulation are given in the following remark.

Remark IV.1. (Cost function)

- 1) If only the good channel is used, that is, $\mathbf{P}_i(s_i = 0) = 1$, then the capacity of the BSSC is equal to zero, because $\mathbf{P}(s_i = 0) = 1$ corresponds to channel input $a_i = b_{i-1}$, a deterministic function for $i = 0, 1, \dots, n$ (this also follows from $I(A_i; B_i | B_{i-1}) \Big|_{A_i=B_{i-1}} = 0, i = 0, 1, \dots, n$). The capacity of the BSSC is also equal to zero if only the bad channel is used $\mathbf{P}_i(s_i = 0) = 0$, for $i = 0, 1, \dots, n$.
- 2) It is shown shortly that the optimal channel input distribution that achieves the unconstrained capacity of the BSSC, corresponds to a fixed occupation of the two states. Upon introducing the transmission cost constraint, one is not allowed to use the state corresponding to the good channel beyond a certain threshold, because the overall cost of transmission needs to be satisfied.
- 3) If $\beta > \alpha \geq 0.5$, then we reverse the transmission cost, while if α and β are less than 0.5, then flip the corresponding channel input probabilities.

A. Capacity of the BSSC with feedback

In this section, we apply Theorem III.1 and Theorem III.2 to calculate the closed form expressions of the capacity achieving channel input distribution, the corresponding channel output distributions, and to show that these are time-invariant. Further, we employ these theorems to calculate the feedback capacity with and without cost constraints.

1) Feedback capacity of the BSSC without transmission cost:

In the next theorem we show that feedback capacity of the BSSC, without cost constraint, is given by a single letter expression and that the optimal input distribution is time invariant.

Theorem IV.1. (*Feedback capacity and time-invariant property of the optimal distributions*)

Consider the BSSC defined by (IV.144) with feedback, without transmission cost. Then the following hold.

- (a) The capacity achieving channel input distribution and the corresponding channel output distribution which maximize the FTFI capacity, $C_{A^n \rightarrow B^n}^{FB,BSSC}$, are time-invariant and given by the following expressions.

$$\pi_i^*(a_i|b_{i-1}) = \pi^{TI}(a_i|b_{i-1}) = \begin{bmatrix} v & 1-v \\ 1-v & v \end{bmatrix}, \quad \forall i \in \{0, 1, \dots, n\} \quad (\text{IV.151})$$

$$\mathbf{P}_i^{\pi^*}(b_i|b_{i-1}) = \mathbf{P}^{TI}(b_i|b_{i-1}) = \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix}, \quad \forall i \in \{0, 1, \dots, n\} \quad (\text{IV.152})$$

where

$$\lambda = \frac{1}{1+2^\mu}, \quad \mu = \frac{H(\beta) - H(\alpha)}{1 - \alpha - \beta}, \quad v = \frac{1 - (1 - \beta)(1 + 2^\mu)}{(\alpha + \beta - 1)(1 + 2^\mu)}. \quad (\text{IV.153})$$

Moreover,

$$C_{A^n \rightarrow B^n}^{FB,BSSC} = (n+1) \max_{\pi^{TI}(a_0|b_{-1})} I(A_0; B_0 | B_{-1} = b_{i-1}), \quad \forall b_{i-1} \in \{0, 1\} \quad (\text{IV.154})$$

$$= (n+1) [H(\lambda) - vH(\alpha) - (1-v)H(\beta)]. \quad (\text{IV.155})$$

- (b) The feedback capacity is given by

$$C_{A^\infty \rightarrow B^\infty}^{FB,BSSC} = \max_{\pi^{TI}(a_0|b_{-1})} I(A_0; B_0 | B_{-1} = b_{i-1}), \quad \forall b_{i-1} \in \{0, 1\} \quad (\text{IV.156})$$

$$= H(\lambda) - vH(\alpha) - (1-v)H(\beta) \quad (\text{IV.157})$$

and it is independent of the initial state.

Proof: The proof of Theorem. IV.1 is given in Appendix C. ■

Theorem IV.1, specifically (IV.154), illustrates the non-nested and time-invariant property, which gives a direct connection of the BSSC and memoryless channels. Note, that these properties hold due to the “symmetric” form of the BSSC. As will show at the end of the current section via simulations, the time-invariant property does not hold for general Binary Unit Memory Channel (BUMC).

The BSSC without cost constraint is equivalent to the POST channel investigated in [17]. The authors in [17] derived an expression for feedback capacity, which is equivalent to (IV.156), by using the convex hull theorem. Theorem IV.1 compliments the results in [17] in the sense that it provides closed form expressions of the capacity achieving distribution and the corresponding optimal channel output conditional distribution. More importantly, it shows that these distributions are time-invariant and correspond

to the non-nested optimization problem (IV.154), which is directly analogous to Shannon's two-letter capacity formulae of memoryless channels.

The structure of our expression (IV.157) provides insight on how the occupancy of the two states affects the capacity. Recall that the state of the channel defines which of the two binary symmetric channels is in use at each time instant. Since $\mathbf{P}_{S_i}(0) = \mathbf{P}_{A_i, B_{i-1}}(0, 0) + \mathbf{P}_{A_i, B_{i-1}}(1, 1)$, then by substituting the capacity achieving input distribution we have $\mathbf{P}_{S_i}(0) = \nu$. Thus, the optimal occupancy, or equivalently the optimal time sharing, among the two binary symmetric channels with crossover probabilities α, β , is given by ν which is a function of the channel parameters α and β . This interpretation is obvious in the feedback capacity expression (IV.157) and this expression is similar to the capacity of the memoryless binary symmetric channel. However, for the BSSC the maximization of the output process corresponds to a time invariant, first order doubly stochastic Markov process.

2) Feedback capacity of the BSSC with transmission cost:

Next, we consider the BSSC with transmission cost constraint defined by (IV.148). Since $C_{A^n \rightarrow B^n}^{FB, BSSC}(\kappa)$ is a convex optimization problem the optimal channel input conditional distribution occurs on the boundary of the constraint, i.e., for $\kappa \geq \kappa_{\max}$ $C_{A^n \rightarrow B^n}^{FB, BSSC}(\kappa)$ is constant and equal to the unconstrained capacity given in Theorem IV.1.

Theorem IV.2. Consider the BSSC defined by (IV.144) with feedback and transmission cost constraint defined by (IV.148). Then the following hold.

- (a) The optimal channel input distribution which corresponds to $C_{A^n \rightarrow B^n}^{FB, BSSC}(\kappa)$ and the optimal output distribution, are time-invariant and given by

$$\pi_i^*(a_i|b_{i-1}) = \pi^{TI}(a_i|b_{i-1}) = \begin{bmatrix} \kappa & 1-\kappa \\ 1-\kappa & \kappa \end{bmatrix}, \quad \forall i = 0, 1, \dots, n \quad (\text{IV.158})$$

$$\mathbf{P}_i^{\pi^*}(b_i|b_{i-1}) = \mathbf{P}^{TI}(b_i|b_{i-1}) = \begin{bmatrix} \bar{\lambda} & 1-\bar{\lambda} \\ 1-\bar{\lambda} & \bar{\lambda} \end{bmatrix}, \quad \forall i = 0, 1, \dots, n \quad (\text{IV.159})$$

where

$$\bar{\lambda} = \alpha\kappa + (1-\kappa)(1-\beta). \quad (\text{IV.160})$$

Moreover,

$$C_{A^n \rightarrow B^n}^{FB, BSSC}(\kappa) = (n+1) \max_{\pi^{TI}(a_0|b_{-1}): \mathbb{E}\{a_i, b_{i-1}\} \leq \kappa} I(A_0; B_0 | B_{-1} = b_{i-1}), \quad \forall b_{i-1} \in \{0, 1\} \quad (\text{IV.161})$$

- (b) The feedback capacity is given by

$$C_{A^n \rightarrow B^n}^{FB, BSSC}(\kappa) = \begin{cases} H(\bar{\lambda}) - \kappa H(\alpha) - (1-\kappa)H(\beta) & \text{if } \kappa \leq \kappa_{\max} \\ H(\lambda) - \kappa_{\max} H(\alpha) - (1-\kappa_{\max})H(\beta) & \text{if } \kappa > \kappa_{\max} \end{cases} \quad (\text{IV.162})$$

where κ_{\max} is equal to ν defined by (IV.153).

This proof is similar to the proof of Theorem IV.1 and is given in Appendix D.

The unconstrained and constrained feedback capacity of the BSSC are depicted in Figure IV.2. In particular, Figure IV.2a depicts the unconstrained capacity of the BSSC for all possible values of the parameters $\alpha, \beta \in [0, 1]$. Figure IV.2b, depicts how the transmission cost affects the capacity of the BSSC for all possible values of the parameters $\alpha, \beta \in [0, 1]$, and for three different choices κ . The inner plot corresponds to the unconstrained case ($\kappa = \kappa_{\max}$).

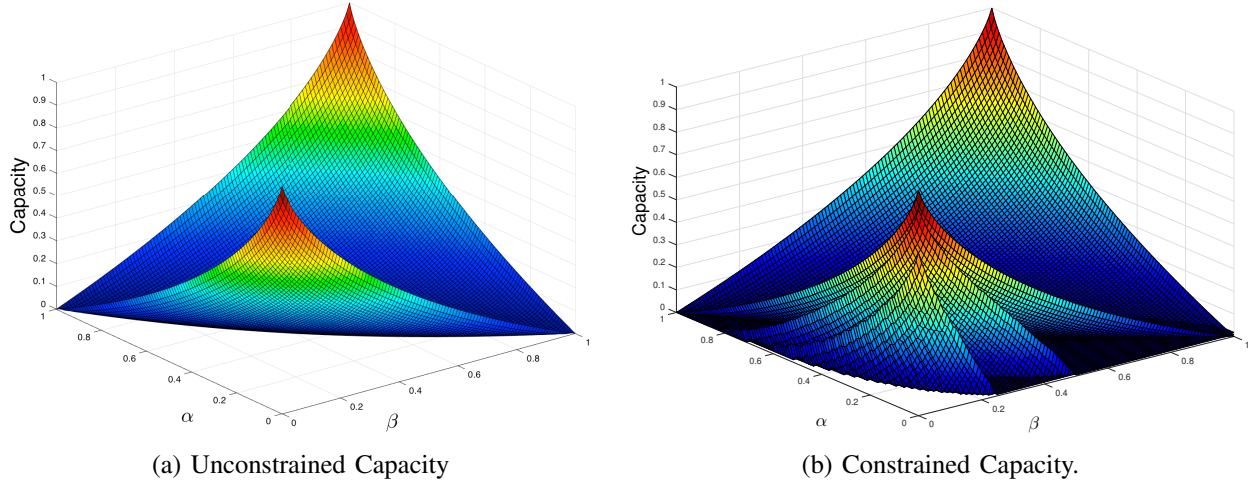


Fig. IV.2: Capacity of BSSC with feedback.

3) Error exponents for the BSSC with feedback:

In this section we apply the results of Section III-D to the BSSC, and we evaluate the error exponent and the probability of error, for the capacity achieving input distribution with feedback denoted by π^{TI} and defined by (IV.151).

It is straightforward to verify that evaluating (III.133) at the capacity achieving input distribution defined (IV.151), this term is independent of the initial state of the channel, and is given by

$$E_{0,n}^{\pi^{TI}}(\rho, b_{-1}) \equiv E_{0,n}^{\pi^{TI}}(\rho). \quad (\text{IV.163})$$

Consequently, the upper bound bound on the probability of error is also independent of the initial state, and is given by

$$\mathbf{P}_{e,m}^{(n)} \leq 4|\mathbb{B}|2^{\{-n[-\rho R + F_n(\rho)]\}}, \quad \forall m \in \mathcal{M}_n, \quad 0 \leq \rho \leq 1. \quad (\text{IV.164})$$

Moreover, since the capacity achieving distribution is time invariant, then $\Lambda_i^\pi(s_i, s_{i-1}) = \Lambda^{\pi^{TI}}(s_i, s_{i-1})$: $i = 0, 1, \dots, n$. Then, by substituting the time invariant capacity achieving distribution and the channel distribution in (III.139), we obtain

$$\Lambda^{\pi^{TI}}(0, 0) = \Lambda^{\pi^{TI}}(1, 1) = \left[v\alpha^{\frac{1}{1+\rho}} + (1-v)(1-\beta)^{\frac{1}{1+\rho}} \right]^{1+\rho} \quad (\text{IV.165})$$

$$\Lambda^{\pi^{TI}}(0, 1) = \Lambda^{\pi^{TI}}(1, 0) = \left[v(1-\alpha)^{\frac{1}{1+\rho}} + (1-v)\beta^{\frac{1}{1+\rho}} \right]^{1+\rho} \quad (\text{IV.166})$$

The largest eigenvalue for the resulted 2×2 Toeplitz matrix matrix and the ratio of the maximum and minimum components of the positive eigenvector that correspond to the largest eigenvalue are given by

$$\begin{aligned} \lambda_{\max}^{\pi^{TI}}(\rho) &= \Lambda(0, 0) + \Lambda(0, 1) \\ &= \left[v\alpha^{\frac{1}{1+\rho}} + (1-v)(1-\beta)^{\frac{1}{1+\rho}} \right]^{1+\rho} + \left[v(1-\alpha)^{\frac{1}{1+\rho}} + (1-v)\beta^{\frac{1}{1+\rho}} \right]^{1+\rho}. \end{aligned} \quad (\text{IV.167})$$

$$\frac{v_{\max}}{v_{\min}} = 1. \quad (\text{IV.168})$$

Substituting (IV.167) and (IV.168) in (III.143) we obtain

$$E_0^{\pi^{TI}}(\rho) = -\log \lambda_{\max}^{\pi^{TI}}(\rho) \quad (\text{IV.169})$$

$$F_\infty(\rho) \triangleq \lim_{n \rightarrow \infty} F_n(\rho) = -\log \lambda_{\max}^{\pi^{TI}}(\rho). \quad (\text{IV.170})$$

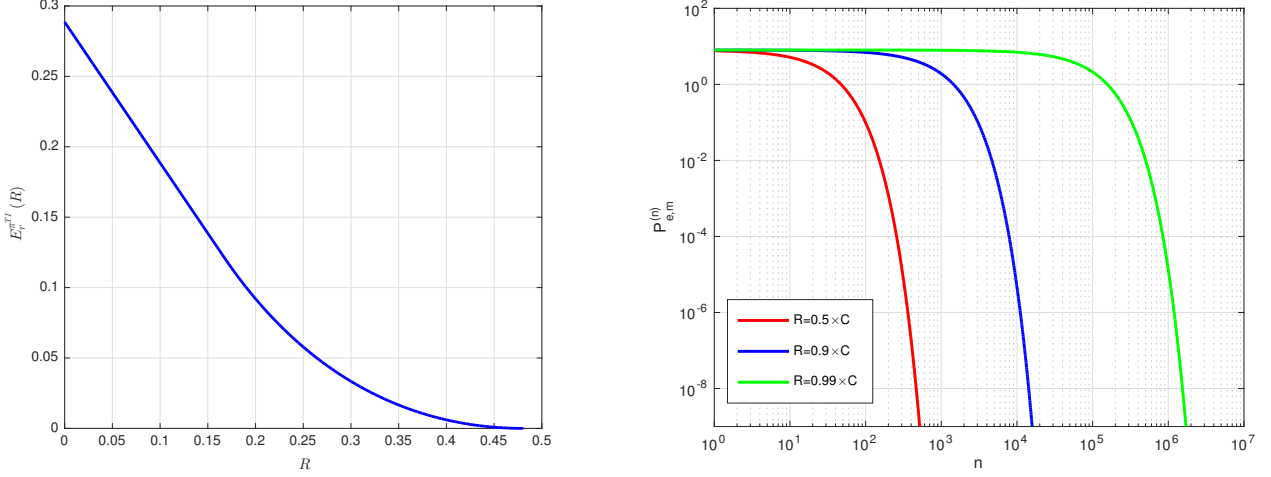


Fig. IV.3: (a) Error exponent and (b) Probability of error for the BSSC with parameters $\alpha = 0.95$, $\beta = 0.8$.

Then, by definition

$$E_r^{\pi^{TI}}(R) \triangleq \max_{0 \leq \rho \leq 1} \{F_\infty(\rho) - \rho R\}. \quad (\text{IV.171})$$

Hence, the probability of error is given by

$$\mathbf{P}_{e,m}^{(n)} \leq 4 \times |2| \times 2^{\left\{ -n \left[-\rho R - \log \lambda_{\max}^{\pi^{TI}}(\rho) \right] \right\}}. \quad (\text{IV.172})$$

Better bounds can be obtained if both the encoder and the decoder know the initial state of the channel. In this case the cardinality of the state, $|2|$, is omitted from (IV.172) [Problem 5.37, [25]]. The error exponent and the probability of error, optimized with respect to ρ , are given in Fig. IV.3. Obviously, even better bounds can be obtained by optimizing with respect to the channel input distribution. However, even for DMC's, the error exponent which is analogue to (IV.171), is often evaluated at the capacity achieving distribution of the ergodic capacity.

B. Capacity without feedback of the BSSC

In this section we apply Theorem II.2, to show that the feedback capacity of the BSSC is achieved by a time invariant first order channel input distribution without feedback.

Theorem IV.3. (Capacity of BSSC without Feedback with & without Transmission Cost)

Consider the BSSC defined by (IV.144) without feedback. Then the following hold.

- (a) For a channel with transmission cost constraint defined by (IV.148), the optimal channel input distribution which corresponds to $C_{A^n, B^n}^{\text{noFB, BSSC}}(\kappa)$ is time-invariant first-order Markov, and it is given by

$$\mathbf{P}_i^{\text{noFB},*}(a_i|a_{i-1}) = \pi^{\text{noFB, TI}}(a_i|a_{i-1}) = \begin{bmatrix} \frac{1 - \kappa - \sigma}{1 - 2\sigma} & \frac{\kappa - \sigma}{1 - \kappa - \sigma} \\ \frac{\kappa - \sigma}{1 - 2\sigma} & \frac{1 - 2\sigma}{1 - \kappa - \sigma} \end{bmatrix}, \quad i = 1, 2, \dots, n, \quad (\text{IV.173})$$

where $\sigma = \alpha\kappa + \beta(1 - \kappa)$. Moreover (IV.173) induces the optimal channel input and channel output distributions $\pi^{TI}(a_i|b_{i-1})$ and $\mathbf{P}^{TI}(b_i|b_{i-1})$ of the BSSC with feedback and transmission cost.

- (b) For a channel without transmission cost (a) holds with $\kappa = \kappa^*$ and $\sigma = \sigma^* = \alpha\kappa^* + \beta(1 - \kappa^*)$.

(c) The capacity the BSSC without feedback and transmission cost is given by

$$C_{A^n, B^n}^{noFB, BSSC} = (n+1) \max_{\pi^{noFB, TI}(a_1|a_0)} I(A_1; B_1|B_0) = (n+1) C_{A^\infty \rightarrow B^\infty}^{FB, BSSC} \quad (IV.174)$$

and similarly, if there is a transmission cost.

Proof: (a) By applying Theorem II.2, it suffices to show that there exists an input distribution without feedback which induces the capacity achieving channel input distribution with feedback, $\pi_i^*(a_i|b_{i-1})$. For the BSSC, it is clear that, if any input distribution without feedback induces $\pi_i^*(a_i|b_{i-1}) = \pi^{TI}(a_i|b_{i-1})$ given by (IV.158), then it also induces the optimal output process $\mathbf{P}_i^{\pi^{noFB, TI}, *}(b_i|b_{i-1}) = \mathbf{P}^{TI}(b_i|b_{i-1})$ given by (IV.159), since

$$\mathbf{P}^{TI}(b_i|b_{i-1}) = \sum_{a_i \in \{0,1\}} \mathbf{P}(b_i|a_i, b_{i-1}) \pi^{TI}(a_i|b_{i-1}). \quad (IV.175)$$

Suppose the distribution of the initial state b_{-1} is given by the stationary distribution of the output process, that is, $\mathbf{P}_{b_{-1}}(0) = \mathbf{P}_{b_{-1}}(1) = 0.5$. Then, we show by induction that there exist a time invariant, first order Markov channel input distribution without feedback that induces the time invariant channel input distribution with feedback. For $i = 0$, the optimal channel input distribution without feedback is equal to optimal channel input distribution with feedback, that is, $\pi_0^{noFB}(a_0|b_{-1}) = \pi^{TI}(a_0|b_{-1})$, and is given by (IV.158), since b_{-1} is the initial state known at the encoder. Therefore, the corresponding channel output distribution with feedback, $\mathbf{P}_0^{\pi^*}(b_0|b_{-1})$, is induced and since $\mathbf{P}_0^{\pi^*}(b_0|b_{-1}) = \mathbf{P}^{TI}(b_0|b_{-1})$ is doubly stochastic, then $\mathbf{P}_0^*(b_0 = 0) = \mathbf{P}_0^*(b_0 = 1) = 0.5$.

For $i = 1$, the following identities hold, in general.

$$\begin{aligned} \mathbf{P}_1(a_1|b_0) &= \sum_{a_0 \in \{0,1\}} \mathbf{P}_1(a_1|a_0, b_0) \mathbf{P}_0(a_0|b_0) \\ &= \sum_{a_0 \in \{0,1\}} \mathbf{P}_1(a_1|a_0, b_0) \frac{\mathbf{P}_0(b_0, a_0)}{\mathbf{P}_0(b_0)} \\ &= \sum_{a_0 \in \{0,1\}} \frac{\mathbf{P}_1(a_1|a_0, b_0)}{\mathbf{P}_0(b_0)} \sum_{b_{-1} \in \{0,1\}} \mathbf{P}(b_0|a_0, b_{-1}) \mathbf{P}_0(a_0|b_{-1}) \mathbf{P}(b_{-1}) \end{aligned} \quad (IV.176)$$

Next using (IV.176), we investigate whether there exists a first order Markov channel input distribution without feedback, $\mathbf{P}_1(a_1|a_0, b_0) = \pi_1^{noFB}(a_1|a_0)$, which induces the time-invariant capacity achieving input distribution with feedback, $\pi^{TI}(a_1|b_0)$, given by (IV.158). Therefore, we need to determine whether the following identity holds for some $\pi_1^{noFB}(a_1|a_0)$. From (IV.176),

$$\pi^{TI}(a_1|b_0) \stackrel{?}{=} \sum_{a_0 \in \{0,1\}} \frac{\pi_1^{noFB}(a_1|a_0)}{\mathbf{P}_0^*(b_0)} \sum_{b_{-1} \in \{0,1\}} \mathbf{P}(b_0|a_0, b_{-1}) \pi^{TI}(a_0|b_{-1}) \mathbf{P}(b_{-1}) \quad (IV.177)$$

Note that $\mathbf{P}_0(a_0|b_{-1}) = \pi^{TI}(a_0|b_{-1})$ and $\mathbf{P}_0(b_0) = \mathbf{P}_0^*(b_0)$ hold due to step $i = 0$. Solving the system of resulting equations, yields that there exists a channel input distribution without feedback, defined by (IV.173), that induces $\pi^{TI}(a_1|b_0)$. Therefore, it also induces the time invariant optimal output distribution, $\mathbf{P}_i^{\pi^*}(b_1|b_0) = \mathbf{P}^{TI}(b_1|b_0)$ given by (IV.159), and its corresponding optimal marginal distribution $\mathbf{P}^*(b_0)$.

Next, suppose that for time up to time $i = j - 1$, the first order Markov input distribution defined by (IV.173) induces the time invariant capacity achieving distribution with feedback, $\{\pi^{TI}(a_i|b_{i-1}) : i = 2, 3, \dots, j - 1\}$, given by (IV.158), and therefore it induces, $\{\mathbf{P}^{TI}(b_i|b_{i-1}) : i = 2, 3, \dots, j - 1\}$ given by (IV.159), and its corresponding optimal marginal distribution $\{\mathbf{P}^*(b_i) : i = 2, 3, \dots, j - 1\}$. Then, at time

$i = j$, the following identity holds.

$$\begin{aligned}
\mathbf{P}_j(a_j|b_{j-1}) &= \sum_{a^{j-1}, b^{j-2}} \mathbf{P}_j(a_j|a^{j-1}, b^{j-1}) \mathbf{P}_{j-1}(a^{j-1}, b^{j-2}|b_{j-1}) \\
&= \sum_{a^{j-1}, b^{j-2}} \frac{\mathbf{P}_j(a_j|a^{j-1}, b^{j-1})}{\mathbf{P}(b_{j-1})} \mathbf{P}(b_{j-1}|a^{j-1}, b^{j-2}) \mathbf{P}(a_{j-1}|a^{j-2}, b^{j-2}) \mathbf{P}(a^{j-2}, b^{j-2}) \\
&= \sum_{a^{j-1}, b^{j-2}} \frac{\mathbf{P}_j(a_j|a^{j-1}, b^{j-1})}{\mathbf{P}^*(b_{j-1})} \mathbf{P}(b_{j-1}|a_{j-1}, b_{j-2}) \pi^{TI}(a_{j-1}|b_{j-2}) \mathbf{P}^*(a^{j-2}, b^{j-2}). \quad (\text{IV.178})
\end{aligned}$$

The last equality holds since the distributions $\mathbf{P}^*(b_{j-1})$, $\pi^{TI}(a_{j-1}|b_{j-2})$, $\mathbf{P}^*(a^{j-2}, b^{j-2})$ were induced from the previous steps $i = 0, 1, \dots, j-1$. Subsequently, we investigate whether there exists a first order Markov channel input distribution, $\mathbf{P}_j(a_j|a^{j-1}, b^{j-1}) = \pi_j^{noFB}(a_j|a_{j-1})$, that satisfies (IV.178). That is,

$$\begin{aligned}
\pi^*(a_j|b_{j-1}) &\stackrel{?}{=} \sum_{a^{j-1}, b^{j-2}} \frac{\pi_j^{noFB}(a_j|a_{j-1})}{\mathbf{P}^*(b_{j-1})} \mathbf{P}(b_{j-1}|a_{j-1}, b_{j-2}) \pi^{TI}(a_{j-1}|b_{j-2}) \mathbf{P}^*(a^{j-2}, b^{j-2}) \\
&= \sum_{a_{j-1}} \frac{\pi_j^{noFB}(a_j|a_{j-1})}{\mathbf{P}^*(b_{j-1})} \sum_{b_{j-2}} \mathbf{P}(b_{j-1}|a_{j-1}, b_{j-2}) \pi^{TI}(a_{j-1}|b_{j-2}) \sum_{a^{j-2}, b^{j-3}} \mathbf{P}^*(a^{j-2}, b^{j-2}) \\
&= \sum_{a_{j-1}} \frac{\pi_j^{noFB}(a_j|a_{j-1})}{\mathbf{P}^*(b_{j-1})} \sum_{b_{j-2}} \mathbf{P}(b_{j-1}|a_{j-1}, b_{j-2}) \pi^*(a_{j-1}|b_{j-2}) \mathbf{P}^*(b_{j-2}) \quad (\text{IV.179})
\end{aligned}$$

Solving, the system of equation resulting from equation (IV.179), yields the time-invariant first order Markov input distribution defined by (IV.173). Since, the time invariant first order Markov channel input distribution without feedback defined by (IV.173), induces the optimal channel input distribution with feedback $\forall i = 1, 2, \dots, j$, then it is the time invariant capacity achieving input distribution without feedback.

(b) Holds since for the BSSC without transmission cost $\kappa = \kappa^*$, and therefore $\sigma = \sigma^* = \alpha \kappa^* + \beta(1 - \kappa^*)$.
(c) Since, $\{\pi^{noFB, TI}(a_i|a_{i-1}) \equiv \pi^{noFB, TI}(a_i|a_0) : i = 1, 2, \dots, n\}$ induces $\{\pi^{TI}(a_i|b_{i-1}) : i = 1, 2, \dots, n\}$ given by (IV.158), and $\{\mathbf{P}^{TI}(b_i|b_{i-1}) : i = 1, 2, \dots, n\}$ given by (IV.159), then the channel capacity without feedback and transmission cost is given by (IV.174). Similarly, for the constrained capacity we have $C_{A^n; B^n}^{noFB, BSSC}(\kappa) = C_{A^\infty \rightarrow B^\infty}^{FB, BSSC}(\kappa)$. ■

C. Special cases of the BSSC

1) Memoryless BBSC ($\alpha = \beta = 1 - \varepsilon$, $\varepsilon \neq 0.5$):

Consider the trivial case where $\alpha = \beta \triangleq 1 - \varepsilon$. Then, given the state $s_i = a_i \oplus b_{i-1}$, the BSSC degenerates to the Discrete Memoryless - Binary Symmetric Channel (DM-BSC) with cross over probability ε . By employing (IV.151)-(IV.156) and (IV.173), then $\mu = 0$ and $\nu = \lambda = 0.5$, the capacity achieving input distribution and the corresponding output distribution are memoryless and uniformly distributed, and the capacity expression reduces to

$$C^{DM-BSC} = H((1 - \varepsilon)(1 - 0.5) + \varepsilon 0.5) - 0.5H(\varepsilon) - 0.5H(\varepsilon) = 1 - H(\varepsilon).$$

This are the known results of the memoryless BSC.

2) Best and Worst BBSC ($\alpha = 1$, $\beta = 0.5$):

Consider the case $\alpha = 1$ and $\beta = 0.5$. This channel decomposes to a noiseless BSC channel with crossover probability 0 if $s_i = a_i \oplus b_{i-1} = 0$, and to a noisy BSC channel with crossover probability 0.5 if $s_i = a_i \oplus b_{i-1} = 1$. By invoking (IV.151)-(IV.156), then $\nu = 0.6$, $\lambda = 0.8$, the channel capacity is equal to

$$C^{FB, BSSC} \Big|_{\alpha=1, \beta=0.5} = C^{noFB, BSSC} \Big|_{\alpha=1, \beta=0.5} = H(0.2) - 0.6H(1) - 0.4H(0.5) = 0.3219$$

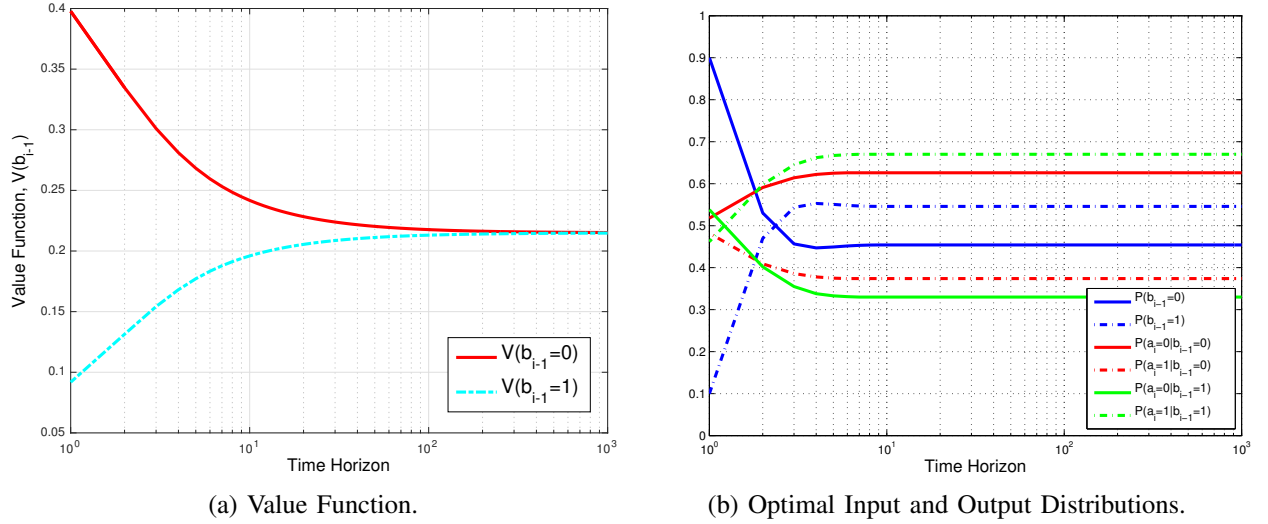


Fig. IV.4: Unconstrained *BIBO-UMCO* channel with parameters $\alpha_1 = 0.9$, $\alpha_2 = 0.2$, $\alpha_3 = 0.1$ and $\alpha_4 = 0.4$.

the optimal channel input distributions with and without feedback are given by

$$\pi^{TI}(a_i|b_{i-1}) = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}, \quad \pi^{noFB,TI}(a_i|a_{i-1}) = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

and the optimal channel output distribution for both is given by

$$\mathbf{P}^{TI}(b_i|b_{i-1}) = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}.$$

This completes the analysis of degenerate BSSC.

D. Capacity of the Binary Input Binary Output - Unit Memory Channel Output (*BIBO-UMCO*) channel with feedback

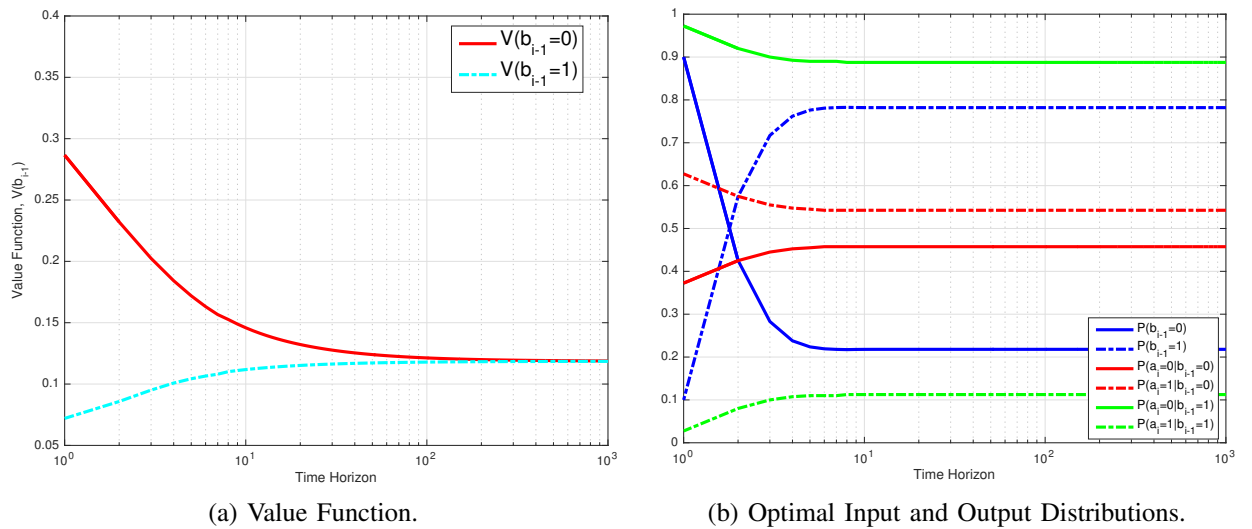


Fig. IV.5: Constrained *BIBO-UMCO* channel with parameters $\alpha_1 = 0.9$, $\alpha_2 = 0.2$, $\alpha_3 = 0.1$, $\alpha_4 = 0.4$ and $k = 0.1877$.

In this section, we employ the dynamic programming results obtained in Section III-A, to calculate the

feedback capacity of *BIBO-UMCO* channel, denoted by

$$\mathbf{P}(b_i|b_{i-1}, a_i) = \begin{matrix} & \begin{matrix} 00 & 01 & 10 & 11 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 1-\alpha_1 & 1-\alpha_2 & 1-\alpha_3 & 1-\alpha_4 \end{pmatrix} \end{matrix}, \quad i = 0, \dots, n \quad (\text{IV.180})$$

with and without transmission cost. In addition, we calculate the capacity achieving input distributions with feedback and the respective optimal output distributions.

1) *Without cost constraint:* Consider the *BIBO-UMCO* channel (IV.180) with parameters $\alpha_1 = 0.9$, $\alpha_2 = 0.2$, $\alpha_3 = 0.1$ and $\alpha_4 = 0.4$. By employing dynamic programming equations (III.67)-(III.68) the convergence of the value functions without transmission cost, and the convergence of the optimal input distributions with feedback and the corresponding output distributions are depicted in Figures IV.4a and IV.4b, respectively. To characterize the feedback capacity and the capacity achieving input distribution of the BIBO-UMCO channel we employ Algorithm 1, which yields the following results.

$$\pi^\infty(a_i|b_{i-1}) = \begin{bmatrix} 0.626 & 0.33 \\ 0.374 & 0.67 \end{bmatrix}, \quad C^{FB, \text{BIBO-UMCO}} = 0.215 \text{ bits/per channel use.}$$

2) *With cost constraint.:* Consider the *BIBO-UMCO* channel (IV.180) with parameters $\alpha_1 = 0.9$, $\alpha_2 = 0.2$, $\alpha_3 = 0.1$, $\alpha_4 = 0.4$ and $k = 0.1877$. By employing dynamic programming equations (III.91)-(III.92), Figures IV.5a and IV.5b depict the convergence of the value functions with transmission cost, and the convergence of the optimal channel input distributions with feedback and the corresponding output distributions.

V. CONCLUSIONS

We apply the dynamic programming recursions and necessary and sufficient conditions for any channel input conditional distribution to achieve capacity, to identify necessary and sufficient conditions such that the nested optimization problem $C_{A^n \rightarrow B^n}^{FB}$ reduces to a non-nested optimization problem. This gives rise to the single letter characterization of feedback capacity. The methodology can be easily generalized to channels that have finite memory on the previous outputs.

These results are applied to the BSSC with feedback, with and without cost constraint, to calculate the feedback capacity, the capacity achieving input distribution, and the corresponding output distribution. One of the fascinating results is that feedback capacity is characterized by a single letter expression that is precisely analogous to the single letter characterization of capacity of DMCs. Additionally, we show that a first order Markov channel input distribution without feedback achieves feedback capacity. We also derive an upper bound on the error probability of maximum likelihood decoding.

APPENDIX A PROOF OF LEMMA. III.1

We can re-write (III.110) as follows.

$$\tilde{V}_t(b_{-1}) + \frac{1}{t} \tilde{V}_t(b_{-1}) = \sup_{\pi^\infty(\cdot|b_{-1})} \left\{ \sum_{a_0} \left\{ \sum_{b_0} \log \left(\frac{\mathbf{P}(b_0|b_{-1}, a_0)}{\mathbf{P}\pi^\infty(b_0|b_{-1})} \right) \mathbf{P}(b_0|b_{-1}, a_0) \right. \right. \quad (\text{A.181})$$

$$\left. + \sum_{b_0} \left(\tilde{V}_{t-1}(b_0) + \frac{1}{t} \tilde{V}_t(b_{-1}) \right) \mathbf{P}(b_0|b_{-1}, a_0) \right\} \pi^\infty(a_0|b_{-1}). \quad (\text{A.182})$$

Assumptions III.2, imply that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \tilde{V}_t(b_{-1}) = J^*, \quad \forall b_{-1} \in \mathbb{B} \quad (\text{A.183})$$

and that the limit does not depend on $b_{-1} \in \mathbb{B}$. Moreover, under Assumption III.2, (A.183), taking the limit of both sides of (A.182), the following dynamic programming equation is obtained.

$$J^* + v(b_{-1}) = \lim_{t \rightarrow \infty} \left\{ \frac{1}{t} \tilde{V}_t(b_{-1}) + \left(\tilde{V}_t(b_{-1}) - tJ^* \right) \right\} \quad (\text{A.184})$$

$$\stackrel{(a)}{=} \lim_{t \rightarrow \infty} \sup_{\pi^\infty(\cdot|b_{-1})} \left\{ \sum_{a_0} \left\{ \sum_{b_0} \log \left(\frac{\mathbf{P}(b_0|b_{-1}, a_0)}{\mathbf{P}^{\pi^\infty}(b_0|b_{-1})} \right) \mathbf{P}(b_0|b_{-1}, a_0) \right. \right. \quad (\text{A.185})$$

$$\left. + \sum_{b_0} \left(\tilde{V}_{t-1}(b_0) - (t-1)J^* + \frac{1}{t} \tilde{V}_t(b_{-1}) - J^* \right) \mathbf{P}(b_0|b_{-1}, a_0) \right\} \pi^\infty(a_0|b_{-1}) \quad (\text{A.186})$$

where (a) is due to (A.181). Since the channel input and output alphabet spaces are at most countable, then we can interchange of the limit and the maximization operations, to obtain dynamic programming equation (III.112).

APPENDIX B PROOF OF THEOREM. III.4

For any $\{\pi^\infty(a_i|b_{i-1}) : i = 0, \dots, n\}$, (III.104) is expressed as follows.

$$J(\pi^\infty, \mu) = \liminf_{n \rightarrow \infty} \frac{1}{n} \mathbf{E}_\mu^{\pi^\infty} \left\{ \sum_{i=0}^{n-1} \ell(b_{i-1}, a_i) \right\} = \liminf_{n \rightarrow \infty} \frac{1}{n} \mathbf{E}_\mu^{\pi^\infty} \left\{ \sum_{i=0}^{n-1} \ell(b_{i-1}, \pi(b_{i-1})) \right\}, \quad \forall \mu(b_{-1}) \in \mathcal{M}(\mathbb{B}) \quad (\text{B.187})$$

$$= \liminf_{n \rightarrow \infty} \frac{1}{n} \mu^T \left(\sum_{i=0}^{n-1} \mathbf{P}(\pi^\infty)^i \right) \ell(\pi^\infty). \quad (\text{B.188})$$

Following [31], it can be shown that the above limit exists but it may depend on the distribution $\mu(\cdot)$ of B_{-1} . However, if $\mathbf{P}(\pi^\infty)$ is irreducible then

$$J(\pi^\infty, \mu^\infty) = \mu^T \mathbf{P}_1(\pi^\infty) \ell(\pi^\infty) = v(\pi^\infty)^T \ell(\pi^\infty) \quad (\text{B.189})$$

where $\mathbf{P}_1(\pi^\infty)$ is the limiting matrix (this follows by the Cesaro limit), and $v(\pi^\infty)$ is the unique invariant probability distribution, which satisfies $\mathbf{P}(\pi^\infty)v(\pi^\infty) = v(\pi^\infty)$. From (B.189), it follows that $J(\pi^\infty, \mu) \equiv J(\pi^\infty)$, that is, it does not depend on the initial distribution μ of B_{-1} . It can be shown that if for all stationary Markov channel input distributions π^∞ the transition matrix $\mathbf{P}(\pi^\infty)$ is irreducible, there exists a solution $V : \mathbb{B} \mapsto \mathbb{R}^{|\mathbb{B}|}$ and $J \in \mathbb{R}$, which satisfies (III.124).

APPENDIX C PROOF OF THEOREM. IV.1

(a) First, we employ the necessary and sufficient conditions of Theorem III.1, to calculate the optimal input and output distributions and the value function at the terminal time. To show the time-invariant property it is sufficient to prove the the value function of the terminal condition, $V_n(b_{n-1})$, is independent of b_{n-1} (part (b) of Theorem III.2). By Theorem III.1, we have

$$V_n(b_{n-1}) = \sum_{b_n} \log \left(\frac{\mathbf{P}_n(b_n|a_n, b_{n-1})}{\mathbf{P}_n^\pi(b_n|b_{n-1})} \right) \mathbf{P}(b_n|a_n, b_{n-1}), \quad \forall a_n \in \mathbb{A}_n \text{ if } \pi_n(a_n|b_{n-1}) \neq 0 \quad (\text{C.190})$$

For $b_{n-1} = 0$ & $a_n = 0$, we obtain

$$\begin{aligned} V_n(b_{n-1} = 0) &= \sum_{b_n} \log \left(\frac{\mathbf{P}_n(b_n|a_n = 0, b_{n-1} = 0)}{\mathbf{P}_n^\pi(b_n|b_{n-1} = 0)} \right) \mathbf{P}(b_n|a_n = 0, b_{n-1} = 0) \\ &= \alpha \log \frac{1 - \mathbf{P}_n^\pi(b_n = 0|b_{n-1} = 0)}{\mathbf{P}_n^\pi(b_n = 0|b_{n-1} = 0)} + \log \frac{1}{1 - \mathbf{P}_n^\pi(b_n = 0|b_{n-1} = 0)} - H(\alpha). \end{aligned} \quad (\text{C.191})$$

For $b_{n-1} = 0$ & $a_n = 1$, we obtain

$$\begin{aligned} V_n(b_{n-1} = 0) &= \sum_{b_n} \log \left(\frac{\mathbf{P}_n(b_n|a_n = 1, b_{n-1} = 0)}{\mathbf{P}_n^\pi(b_n|b_{n-1} = 0)} \right) \mathbf{P}(b_n|a_n = 1, b_{n-1} = 0) \\ &= (1 - \beta) \log \frac{1 - \mathbf{P}_n^\pi(b_n = 0|b_{n-1} = 0)}{\mathbf{P}_n^\pi(b_n = 0|b_{n-1} = 0)} + \log \frac{1}{1 - \mathbf{P}_n^\pi(b_n = 0|b_{n-1} = 0)} - H(\beta). \end{aligned} \quad (\text{C.192})$$

By (C.190), we equate (C.191) and (C.192), to deduce

$$\mathbf{P}_n^\pi(b_n = 0|b_{n-1} = 0) = \lambda = \frac{1}{1 + 2\mu} \quad (\text{C.193})$$

where λ and μ are given in (IV.153). We repeat the above procedure for the pair $b_{n-1} = 1, a_n = 0$ and $b_{n-1} = 1, a_n = 1$, to deduce

$$\mathbf{P}_n^\pi(b_n = 1|b_{n-1} = 1) = \frac{1}{1 + 2\mu} \equiv \lambda. \quad (\text{C.194})$$

Therefore the optimal transition probability of the output process at time n , is given by the doubly stochastic matrix (IV.152). Next, we show that the value function, $V_n(b_{n-1})$, is independent of b_{n-1} . The value function for $b_{n-1} = 1$ and $a_n = 1$ is obtained as follows.

$$\begin{aligned} V_n(b_{n-1} = 1) &= \sum_{b_n} \log \left(\frac{\mathbf{P}_n(b_n|a_n = 1, b_{n-1} = 1)}{\mathbf{P}_n^\pi(b_n|b_{n-1} = 1)} \right) \mathbf{P}(b_n|a_n = 1, b_{n-1} = 1) \\ &= \alpha \log \frac{1 - \mathbf{P}_n^\pi(b_n = 1|b_{n-1} = 1)}{\mathbf{P}_n^\pi(b_n = 1|b_{n-1} = 1)} + \log \frac{1}{1 - \mathbf{P}_n^\pi(b_n = 1|b_{n-1} = 1)} - H(\alpha) \\ &= \alpha \log \frac{1 - \mathbf{P}_n^\pi(b_n = 0|b_{n-1} = 0)}{\mathbf{P}_n^\pi(b_n = 0|b_{n-1} = 0)} + \log \frac{1}{1 - \mathbf{P}_n^\pi(b_n = 0|b_{n-1} = 0)} - H(\alpha). \\ &= V_n(b_{n-1} = 0) \end{aligned} \quad (\text{C.195})$$

Since the value function, $V_n(b_{n-1})$, is independent of b_{n-1} , we apply Theorem III.2.(b), to deduce that the optimal channel input and channel output conditional distributions are time invariant. The optimal channel input conditional distribution is calculated via the expression $\mathbf{P}_n^\pi(b_n|b_{n-1}) = \sum_{A_i} \mathbf{P}_n(b_n|a_n, b_{n-1}) \pi(a_n|b_{n-1})$. For $b_n = 0$ and $b_{n-1} = 0$, we have

$$\begin{aligned} \mathbf{P}_n^\pi(b_n = 0|b_{n-1} = 0) &= \sum_{A_n} \mathbf{P}_n(b_n = 0|a_n, b_{n-1} = 0) \pi(a_n|b_{n-1} = 0) \\ &= \alpha \pi(a_n = 0|b_{n-1} = 0) + (1 - \beta)(1 - \pi(a_n = 0|b_{n-1} = 0)). \end{aligned} \quad (\text{C.196})$$

Solving (C.196) with respect to the input distribution yields

$$\pi(a_n = 0|b_{n-1} = 0) = \frac{1 - (1 - \beta)(1 + 2\mu)}{(\alpha + \beta - 1)(1 + 2\mu)} \equiv \nu. \quad (\text{C.197})$$

Similarly,

$$\begin{aligned} \mathbf{P}_n^\pi(b_n = 1|b_{n-1} = 1) &= \sum_{A_n} \mathbf{P}_n(b_n = 1|a_n, b_{n-1} = 1) \pi(a_n|b_{n-1} = 1) \\ &= \alpha \pi(a_n = 1|b_{n-1} = 0) + (1 - \beta)(1 - \pi(a_n = 0|b_{n-1} = 0)). \end{aligned} \quad (\text{C.198})$$

The above, shows (IV.151). By Theorem III.2.(b), specifically (III.87) evaluated at $t = 0$, we obtain the following expression for the FTFI capacity.

$$\begin{aligned} C_{A^n \rightarrow B^n}^{FB, BSSC} &\stackrel{(\alpha)}{=} \sum_{b_{-1}} V_0(b_{-1}) \mu(b_{-1}) \\ &\stackrel{(\beta)}{=} (n + 1) \max_{\pi(a_0|b_{-1})} \sum_{b_0, a_0, b_{-1}} \left(\frac{\mathbf{P}(b_0|a_0, b_{-1})}{\mathbf{P}^\pi(b_0|b_{-1})} \right) \mathbf{P}(b_0|a_0, b_{-1}) \pi(a_n|b_{n-1}) \mu(b_{-1}), \quad b_{-1} \in \{0, 1\} \\ &\stackrel{(\gamma)}{=} (n + 1) [H(\lambda) - \nu H(\alpha) - (1 - \nu)H(\beta)] \end{aligned} \quad (\text{C.199})$$

where (α) holds by definition (equation (III.69)), (β) holds due to (III.87) evaluated at $t = 0$, (γ) by substituting the time invariant capacity achieving input distribution (IV.151), the corresponding optimal output distribution (IV.152) and any value of $b_{-1} \in \{0, 1\}$.
 (b) holds by definition (equation (II.27)).

APPENDIX D PROOF OF THEOREM. IV.2

(a) By employing the dynamic programming recursion for the constrained problem (III.90) we can show that the value function at the terminal time is independent of b_{n-1} . Therefore, by Theorem III.2, the optimization problem is non-nested and the dynamic programming for the constrained capacity is given by

$$V_i(b_{i-1}) = \sup_{\pi(a_i|b_{i-1}), s \leq 0} \left\{ \sum_{A_n} \sum_{B_n} \log \left(\frac{\mathbf{P}(b_i|b_{i-1}, \alpha_i)}{\mathbf{P}^\pi(b_i|b_{i-1})} \right) \mathbf{P}(b_i|b_{i-1}, \alpha_i) \pi(\alpha_i|b_{i-1}) \right. \\ \left. + s \left\{ \sum_{A_i} \gamma(a_i, b_{i-1}) \pi_n(a_i|b_{i-1}) - \kappa \right\} \right\}, \forall i = 0, 1, \dots, n \quad (\text{D.200})$$

Differentiating (D.200) with respect to the Lagrangian s , we obtain the optimal input distribution of (IV.158). The optimal output distribution is then calculated by $\mathbf{P}_n^\pi(b_n|b_{n-1}) = \sum_{A_n} \mathbf{P}_n(b_n|a_n, b_{n-1}) \pi(a_n|b_{n-1})$ to obtain (IV.159).

(b) Since (i) the optimal input channel conditional distribution and the channel output conditional distribution are time-invariant and (ii) the value function $V_i(b_{i-1})$ is independent of b_{i-1} , $\forall i = 0, 1, \dots, n$, the proof is identical to the proof of Theorem IV.1.(b). The value of κ_{\max} is given when the Lagrangian $s = 0$, i.e. the constrained optimization problem is equivalent to the constrained optimization problem. In this case, $s = 0$, and the optimal channel input conditional distribution for the constrained case is equal to the optimal channel input conditional distribution for the unconstrained case, thus $\kappa|_{s=0} = \kappa_{\max} = \nu$.

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